The Complexity of Planar Boolean #CSP with Complex Weights*

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Abstract. We prove a complexity dichotomy theorem for symmetric complex-weighted Boolean #CSP when the constraint graph of the input must be planar. The problems that are #P-hard over general graphs but tractable over planar graphs are precisely those with a holographic reduction to matchgates. This generalizes a theorem of Cai, Lu, and Xia for the case of real weights. We also obtain a dichotomy theorem for a symmetric arity 4 signature with complex weights in the planar Holant framework, which we use in the proof of our #CSP dichotomy. In particular, we reduce the problem of evaluating the Tutte polynomial of a planar graph at the point $(3,3)$ to counting the number of Eulerian orientations over planar 4-regular graphs to show the latter is #P-hard. This strengthens a theorem by Huang and Lu to the planar setting.

1 Introduction

In 1979, Valiant [2] defined the class #P to explain the apparent intractability of counting the number of perfect matchings in a graph. Yet over a decade earlier, Kasteleyn [3] gave a polynomial-time algorithm to compute this quantity for planar graphs. This was an important milestone in a decades-long research program by physicists in statistical mechanics to determine what problems the restriction to the planar setting renders tractable [4–10, 3, 11–13]. More recently, Valiant introduced matchgates [14, 15] and *holographic* algorithms [16, 17] that rely on Kasteleyn's algorithm to solve certain counting problems over planar graphs. In a series of papers [18–21], Cai et al. characterized the local constraint functions (which define counting problems) that are representable by matchgates in a holographic algorithm.

From the viewpoint of computational complexity, we seek to understand exactly which intractable problems the planarity restriction enable us to efficiently compute. Partial answers to this question have been given in the context of various counting frameworks [22–25]. In every case, the problems that are #P-hard over general graphs but tractable over planar graphs are essentially those characterized by Cai et al. In this paper, we give more evidence for this phenomenon by extending the results of [23] to the setting of complex-valued constraint functions. This provides the most natural setting to express holographic algorithms and transformations.

* Full version with proofs available at [1].

Our main result is a dichotomy theorem for the framework of counting Constraint Satisfaction Problems (#CSP), but our proof is in a generalized framework called Holant problems [26–29]. We briefly introduce the Holant framework and then explain its main advantages. A set of functions $\mathcal{F}$ defines the problem Holant($\mathcal{F}$). An instance of this problem is a tuple $\Omega = (G, \mathcal{F}, \pi)$ called a *signature grid*, where $G = (V, E)$ is a graph, π labels each $v \in V$ with a function $f_v \in \mathcal{F}$, and f_v maps $\{0,1\}^{\deg(v)}$ to $\mathbb{C}$. We also call the functions in $\mathcal{F}$ *signatures*. An assignment σ for every $e \in E$ gives an evaluation $\prod_{v \in V} f_v(\sigma \mid_{E(v)})$, where $E(v)$ denotes the incident edges of v and $\sigma \mid_{E(v)}$ denotes the restriction of σ to $E(v)$. The counting problem on the instance Ω is to compute

$$\text{Holant}_{\Omega} = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v \left(\sigma \mid_{E(v)} \right). \tag{1}$$

Counting the number of perfect matchings in G corresponds to attaching the Exact-One signature at every vertex of G. A function or signature is called *symmetric* if its output depends only on the Hamming weight of the input. We often denote a symmetric signature by the list of its outputs sorted by input Hamming weight in ascending order. For example, $[0, 1, 0, 0]$ is the Exact-One function on three bits. The output is 1 if and only if the input is 001, 010, or 100, and 0 otherwise.

We consider #CSP, which are also parametrized by a set of functions $\mathcal{F}$. The problem #CSP($\mathcal{F}$) is equivalent to Holant($\mathcal{F} \cup \mathcal{EQ}$), where $\mathcal{EQ} = \{=_1, =_2, \dots\}$ and $(=_k) = [1, 0, \dots, 0, 1]$ is the equality signature of arity k. This explicit role of equality signatures permits a finer classification of problems. For a direct definition of #CSP, see [30].

We often consider a Holant problem over bipartite graphs, which is denoted by Holant($\mathcal{F} \mid \mathcal{G}$), where the sets $\mathcal{F}$ and $\mathcal{G}$ contain the signatures available for assignment to the vertices in each partition. Considering the edge-vertex incidence graph, one can see that Holant($\mathcal{F}$) is equivalent to Holant($=_2 \mid \mathcal{F}$). One powerful tool in this setting is the holographic transformation. Let T be a nonsingular 2-by-2 matrix and define $T\mathcal{F} = \{T^{\otimes \operatorname{arity}(f)} f \mid f \in \mathcal{F}\}$, where $T^{\otimes k}$ is the tensor product of k factors of T. Here we view f as a column vector by listing its values in lexicographical order as in a truth table. Similarly $\mathcal{F}T$ is defined by viewing $f \in \mathcal{F}$ as a row vector. Valiant's Holant theorem [16] states that Holant($\mathcal{F} \mid \mathcal{G}$) is equivalent to Holant($\mathcal{F}T^{-1} \mid T\mathcal{G}$).

Cai, Lu, and Xia gave a dichotomy for complex-weighted Boolean #CSP($\mathcal{F}$) in [28]. Let Pl-#CSP($\mathcal{F}$) (resp. Pl-Holant($\mathcal{F}$)) denote the #CSP (resp. Holant problem) defined by $\mathcal{F}$ when the inputs are restricted to planar graphs. In this paper, we investigate the complexity of Pl-#CSP($\mathcal{F}$) for a set $\mathcal{F}$ of symmetric complex-weighted functions. In particular, we would like to determine which sets become tractable under this planarity restriction. Holographic algorithms with matchgates provide planar tractable problems for sets that are matchgate realizable after a holographic transformation. From the Holant perspective, the signatures in $\mathcal{EQ}$ are always available in #CSP($\mathcal{F}$). By the signature theory of Cai and Lu [21], the Hadamard matrix $H = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ essentially defines the only[1] holographic transformation under which $\mathcal{EQ}$ becomes matchgate realizable. Let $\widehat{\mathcal{F}}$ denote $H\mathcal{F}$ for any set of signatures $\mathcal{F}$. Then $\widehat{\mathcal{EQ}}$ is $\{[1,0],[1,0,1],[1,0,1,0],\dots\}$ while $(=_2)(H^{-1})^{\otimes 2}$ is still $=_2$. Therefore #CSP($\mathcal{F}$) and Holant($\mathcal{F} \cup \mathcal{EQ}$) are equivalent to Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$) by Valiant's Holant theorem.

Our main dichotomy theorem is stated as follows.

Theorem 1. *Let $\mathcal{F}$ be a set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-#CSP($\mathcal{F}$) *is* #P*-hard unless $\mathcal{F}$ satisfies one of the following conditions, in which case it is tractable:*
1. #CSP($\mathcal{F}$) *is tractable (cf. [28]); or*
2. *$\widehat{\mathcal{F}}$ is realizable by matchgates (cf. [21]).*

A more explicit description of the tractable cases can be found in Theorem 19.

In many previous dichotomy theorems for Boolean #CSP($\mathcal{F}$), the proof of hardness began by pinning. The goal of this technique is to realize the constant functions $[1,0]$ and $[0,1]$ and was always achieved by a *nonplanar* reduction. This does not imply the collapse of any complexity classes because the tractable sets for #CSP($\mathcal{F}$) include $[1,0]$ and $[0,1]$. However, $\mathcal{EQ}$ with $\{[1,0],[0,1]\}$ are not simultaneously realizable as matchgates. Therefore, according to our main result, if pinning were possible for Pl-#CSP($\mathcal{F}$), then #P collapses to P! Instead, apply the Hadamard transformation and consider Pl-Holant($\widehat{\mathcal{F}} \cup \widehat{\mathcal{EQ}}$). In this Hadamard basis, pinning becomes possible again since $[1,0]$ and $[0,1]$ are included in every tractable set. Indeed, we prove our pinning result in this Hadamard basis, which is discussed in Section 4.

For Holant problems, it is often important to understand the complexity of the small arity cases first [23, 31, 32]. In [23], Cai, Lu, and Xia gave a dichotomy for Pl-Holant(f) when f is a symmetric arity 3 signature while a dichotomy for Holant(f) when f is a symmetric arity 4 signature was shown in [32]. In the proof of the latter result, most of the reductions were planar. However, the crucial starting point for hardness, namely counting Eulerian orientations (#EO) over 4-regular graphs, was not known to be #P-hard under the planarity restriction. Huang and Lu [31] had recently proved that #EO is #P-hard over 4-regular graphs but left open its complexity when the input is also planar. We show that #EO remains #P-hard over planar 4-regular graphs. The problem we reduce from is the evaluation of the Tutte polynomial of a planar graph at the point (3,3), which has a natural expression in the Holant framework. In addition, we determine the complexity of counting complex-weighted matchings over planar 4-regular graphs. The problem is #P-hard except for the tractable case of counting perfect matchings. With these two ingredients, we obtain a dichotomy for Pl-Holant(f) when f is a symmetric arity 4 signature.

Our main result is a generalization of the dichotomy by Cai, Lu, and Xia [23] for Pl-#CSP($\mathcal{F}$) when $\mathcal{F}$ contains symmetric real-weighted Boolean functions. It is natural to consider complex weights in the Holant framework because surprising equivalences between problems are often discovered via complex holographic

[1] Up to transformations under which matchgates are closed.

transformations, sometimes even between problems using only rational weights. Our proof of hardness for #EO over planar 4-regular graphs in Section 3 is a prime example of this. Extending the range from $\mathbb{R}$ to $\mathbb{C}$ also enlarges the set of problems that can be transformed into the framework.

However, a dichotomy for complex weights is more technically challenging. The proof technique of polynomial interpolation often has infinitely many failure cases in $\mathbb{C}$ corresponding to the infinitely many roots of unity, which prevents a brute force analysis of failure cases as was done in [23]. This increased difficulty requires us to develop new ideas to bypass previous interpolation proofs. In particular, we perform a planar interpolation with a rotationally invariant signature to prove the #P-hardness of #EO over planar 4-regular graphs. For the complexity of counting complex-weighted matchings over planar 4-regular graphs, we introduce the notion of planar pairings to build reductions. We show that every planar 3-regular graph has a planar pairing and that it can be efficiently computed. We also refine and extend existing techniques for application in the new setting, including the recursive unary construction, the anti-gadget technique, compressed matrix criteria, and domain pairing.

2 Preliminaries

The framework of Holant problems is defined for functions mapping any $[q]^k \to \mathbb{F}$ for a finite q and some field $\mathbb{F}$. In this paper, we investigate the complex-weighted Boolean Holant problems, that is, all functions are $[2]^k \to \mathbb{C}$. Technically, functions must take complex algebraic numbers for issues of computability.

A *signature grid* $\Omega = (G, \mathcal{F}, \pi)$ consists of a graph $G = (V, E)$, where each vertex is labeled by a function $f_v \in \mathcal{F}$, and $\pi : V \to \mathcal{F}$ is the labeling. If the graph G is planar, then we call Ω a *planar signature grid*. The Holant problem on instance Ω is to evaluate $\mathrm{Holant}_\Omega = \sum_\sigma \prod_{v \in V} f_v(\sigma \mid_{E(v)})$, a sum over all edge assignments $\sigma : E \to \{0, 1\}$.

A function f_v can be represented by listing its values in lexicographical order as in a truth table, which is a vector in $\mathbb{C}^{2^{\deg(v)}}$, or as a tensor in $(\mathbb{C}^2)^{\otimes \deg(v)}$. We also use f^α to denote the value $f(\alpha)$, where α is a binary string. A function $f \in \mathcal{F}$ is also called a *signature*. A symmetric signature f on k Boolean variables can be expressed as $[f_0, f_1, \ldots, f_k]$, where f_w is the value of f on inputs of Hamming weight w. In this paper, we consider symmetric signatures. Since a signature of arity k must be placed on a vertex of degree k, we can represent a signature of arity k by a labeled vertex with k ordered dangling edges. Throughout this paper, we do not distinguish between these two views.

A Holant problem is parametrized by a set of signatures.

Definition 2. *For a signature set $\mathcal{F}$, define the counting problem* $\mathrm{Holant}(\mathcal{F})$ *as:*
Input: A signature grid $\Omega = (G, \mathcal{F}, \pi)$*;*
Output: Holant_Ω.

The problem $\mathrm{Pl\text{-}Holant}(\mathcal{F})$ is defined similarly using a planar signature grid. The Holant^c framework is the special case of Holant problems when the constant signatures of the domain are freely available. In the Boolean domain, the constant signatures are $[1, 0]$ and $[0, 1]$.

Definition 3. *For signature set $\mathcal{F}$,* $\mathrm{Holant}^c(\mathcal{F})$ *denotes* $\mathrm{Holant}(\mathcal{F} \cup \{[0, 1], [1, 0]\})$.

The problem Pl-$\mathrm{Holant}^c(\mathcal{F})$ is defined similarly. A symmetric signature f of arity n is *degenerate* if there exists a unary signature u such that $f = u^{\otimes n}$. Replacing a signature $f \in \mathcal{F}$ by a constant multiple cf, where $c \neq 0$, does not change the complexity of $\mathrm{Holant}(\mathcal{F})$. It introduces a global factor to Holant_{Ω}. Hence, for two signatures f, g of the same arity, we use $f \neq g$ to mean that these signatures are not equal in the projective space sense, i.e. not equal up to any nonzero constant factor. We denote polynomial time Turing equivalence by $\equiv_T$.

An instance of #CSP($\mathcal{F}$) has the following bipartite view. Create a node for each variable and each constraint. Connect a variable node to a constraint node if the variable appears in the constraint function. This bipartite graph is also known as the *constraint graph*. Under this view, we can see that $\#\mathrm{CSP}(\mathcal{F}) \equiv_T \mathrm{Holant}(\mathcal{F} \mid \mathcal{EQ}) \equiv_T \mathrm{Holant}(\mathcal{F} \cup \mathcal{EQ})$, where $\mathcal{EQ} = \{=_1, =_2, =_3, \dots\}$ is the set of equality signatures of all arities. This equivalence also holds for the planar versions of these frameworks. For the #CSP framework, the following two signature sets are tractable [28].

Definition 4. *A k-ary function $f(x_1, \dots, x_k)$ is* affine *if it has the form $\lambda \chi_{Ax=0} \cdot \sqrt{-1}^{\sum_{j=1}^{n} \langle \alpha_j, x \rangle}$, where $\lambda \in \mathbb{C}$, $x = (x_1, x_2, \dots, x_k, 1)^T$, A is a matrix over $\mathbb{F}_2$, α_j is a vector over $\mathbb{F}_2$, and χ is a 0-1 indicator function such that $\chi_{Ax=0}$ is 1 iff $Ax = 0$. Note that the dot product $\langle \alpha_j, x \rangle$ is calculated over $\mathbb{F}_2$, while the summation $\sum_{j=1}^{n}$ on the exponent of $i = \sqrt{-1}$ is evaluated as a sum mod 4 of 0-1 terms. We use $\mathscr{A}$ to denote the set of all affine functions.*

Definition 5. *A function is of* product type *if it can be expressed as a product of unary functions, binary equality functions* ($[1, 0, 1]$)*, and binary disequality functions* ($[0, 1, 0]$)*. We use $\mathscr{P}$ to denote the set of product type functions.*

In the Holant framework, there are two corresponding signature sets that are tractable. A signature f is $\mathscr{A}$-transformable if there exists a holographic transformation T such that $f \in T\mathscr{A}$ and $[1, 0, 1]T^{\otimes 2} \in \mathscr{A}$. Similarly, a signature f is $\mathscr{P}$-transformable if there exists a holographic transformation T such that $f \in T\mathscr{P}$ and $[1, 0, 1]T^{\otimes 2} \in \mathscr{P}$. These two families are tractable because after a transformation by T, it is a tractable #CSP instance.

Matchgates were introduced by Valiant [14, 15] and are combinatorial in nature. They encode computation as a sum of weighted perfect matchings, which has a polynomial-time algorithm by the work of Kasteleyn [3].

We say a signature is a *matchgate signature* if there is some matchgate that realizes this signature and use $\mathscr{M}$ to denote the set of all matchgate signatures. Lemmas 6.2 and 6.3 in [18] (and the paragraph the follows them) characterize the symmetric signatures in $\mathscr{M}$. Instead of formally stating these two lemmas, we explicitly list all the symmetric signatures in $\mathscr{M}$: For any $\alpha, \beta \in \mathbb{C}$,

1. $[\alpha^n, 0, \alpha^{n-1}\beta, 0, \dots, 0, \alpha\beta^{n-1}, 0, \beta^n]$;

2. $[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$;
3. $[0, \alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n]$;
4. $[0, \alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$.

Roughly speaking, the symmetric matchgate signatures have 0 for every other entry (which is called the *parity condition*), and form a geometric progression with the remaining entries. We also say a signature f is $\mathscr{M}$-transformable if there exists a holographic transformation T such that $f \in T\mathscr{M}$ and $[1,0,1]T^{\otimes 2} \in \mathscr{M}$.

3 Pl-Holant Dichotomy for a Symmetric 4-ary Signature

One of our main results is a dichotomy theorem for Pl-Holant(f) when f is a symmetric arity 4 signature with complex weights, which uses the #P-hardness of counting Eulerian orientations over planar 4-regular graphs in a crucial way. Recall that an orientation of the edges of a graph G is an *Eulerian orientation* if for each vertex v of G, the number of incoming edges of v equals the number of outgoing edges of v.

Counting the number of (unweighted) Eulerian orientations over 4-regular graphs was shown to be #P-hard in Theorem V.10 of [31]. We improve this result by showing that this problem remains #P-hard when the input is also planar. We reduce from the problem of counting weighted Eulerian orientations over medial graphs, which are planar 4-regular graphs (see Section 2 in [33] for a definition). Las Vergnas [34] showed that this problem is equivalent to evaluating the Tutte polynomial at the point (3,3), which is #P-hard for planar graphs [22].

Theorem 6 (Theorem 2.1 in [34]). *Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations of the medial graph H of G. Then*

$$2 \cdot \mathrm{T}(G; 3, 3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)}, \tag{2}$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. the number of vertices in which the edges are oriented "in, out, in, out" in cyclic order.

Our proof also uses two notions from [32].

Definition 7. *The matrix* $M_g = \begin{bmatrix} g^{0000} & g^{0010} & g^{0001} & g^{0011} \\ g^{0100} & g^{0110} & g^{0101} & g^{0111} \\ g^{1000} & g^{1010} & g^{1001} & g^{1011} \\ g^{1100} & g^{1110} & g^{1101} & g^{1111} \end{bmatrix}$ *is the* signature matrix *of an arity 4 signature g. When we present g pictorially, we order the four external edges $ABCD$ counterclockwise. In M_g, the row index bits are ordered AB and the column index bits are ordered DC, in a reverse way. This is for convenience so that the signature matrix of the linking of two arity 4 signatures is the matrix product of the signature matrices of the two signatures.*

Now we can prove our hardness result.

Theorem 8. #EULERIAN-ORIENTATIONS *is* #P*-hard on planar 4-regular graphs.*

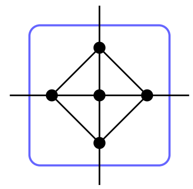

Fig. 1: The planar tetrahedron gadget. Each vertex is assigned $[3,0,1,0,3]$.

Proof. We reduce from calculating the right-hand side of (2) to Pl-Holant$(\neq_2 \mid [0,0,1,0,0])$. The bipartite Holant problem Pl-Holant$(\neq_2 \mid f)$ expresses the right-hand side of (2), where the signature matrix of f is $M_f = \left[\begin{smallmatrix} 0&0&0&1 \\ 0&1&2&0 \\ 0&2&1&0 \\ 1&0&0&0 \end{smallmatrix}\right]$. A holographic transformation by $Z = \left[\begin{smallmatrix} 1&1 \\ i&-i \end{smallmatrix}\right]$ transforms Pl-Holant$(\neq_2 \mid f)$ to Pl-Holant$(\hat{f})$, where the signature matrix of $\hat{f}$ is $M_{\hat{f}} = \left[\begin{smallmatrix} 2&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&2 \end{smallmatrix}\right]$. We also perform a holographic transformation by Z on our target problem Pl-Holant$(\neq_2 \mid [0,0,1,0,0])$ to get Pl-Holant$([3,0,1,0,3])$. Using the planar tetrahedron gadget in Figure 1, we assign $[3,0,1,0,3]$ to every vertex and obtain a gadget with signature $32\hat{g}$, where the signature matrix of $\hat{g}$ is $M_{\hat{g}} = \frac{1}{2}\left[\begin{smallmatrix} 19&0&0&7 \\ 0&7&5&0 \\ 0&5&7&0 \\ 7&0&0&19 \end{smallmatrix}\right]$.

Now we show how to reduce Pl-Holant$(\hat{f})$ to Pl-Holant$(\hat{g})$ by interpolation. Consider an instance Ω of Pl-Holant$(\hat{f})$. Suppose that $\hat{f}$ appears n times in Ω. We construct from Ω a sequence of instances Ω_s of Holant$(\hat{g})$ indexed by $s \geq 1$. We obtain Ω_s from Ω by replacing each occurrence of $\hat{f}$ with the gadget N_s in Figure 2 with $\hat{g}$ assigned to all vertices. Although $\hat{f}$ and $\hat{g}$ are asymmetric signatures, they are invariant under a cyclic permutation of their inputs. Thus, it is unnecessary to specify which edge corresponds to which input. We call such signatures *rotationally symmetric*.

To obtain Ω_s from Ω, we effectively replace $M_{\hat{f}}$ with $M_{N_s} = (M_{\hat{g}})^s$, the sth power of the signature matrix $M_{\hat{g}}$. Let $T = \left[\begin{smallmatrix} 0&0&1&1 \\ 1&1&0&0 \\ -1&1&0&0 \\ 0&0&-1&1 \end{smallmatrix}\right]$. Then $M_{\hat{f}} = T \Lambda_{\hat{f}} T^{-1} = T \left[\begin{smallmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&3 \end{smallmatrix}\right] T^{-1}$ and $M_{\hat{g}} = T \Lambda_{\hat{g}} T^{-1} = T \left[\begin{smallmatrix} 1&0&0&0 \\ 0&6&0&0 \\ 0&0&6&0 \\ 0&0&0&13 \end{smallmatrix}\right] T^{-1}$. We can view our construction of Ω_s as first replacing each $M_{\hat{f}}$ by $T \Lambda_{\hat{f}} T^{-1}$, which does not change the

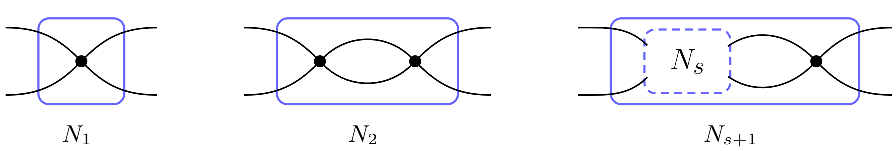

Fig. 2: Recursive construction to interpolate $\hat{f}$. The vertices are assigned $\hat{g}$.

Holant value, and then replacing each $\Lambda_{\hat{f}}$ with $\Lambda_{\hat{g}}^s$. We stratify the assignments in Ω based on the assignment to $\Lambda_{\hat{f}}$. We only need to consider the assignments to $\Lambda_{\hat{f}}$ that assign 0000 j many times, 0110 or 1001 k many times, and 1111 ℓ many times. Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations (including the contributions from T and T^{-1}) on Ω. Then Pl-Holant$_\Omega$ and Pl-Holant$_{\Omega_s}$, for $s \geq 1$, can be expressed as Pl-Holant$_\Omega = \sum_{j+k+\ell=n} 3^\ell c_{jk\ell}$ and Pl-Holant$_{\Omega_s} = \sum_{j+k+\ell=n} (6^k 13^\ell)^s c_{jk\ell}$. This coefficient matrix in the linear system involving Pl-Holant$_{\Omega_s}$ is Vandermonde and of full rank since for any $0 \leq k+\ell \leq n$ and $0 \leq k'+\ell' \leq n$ such that $(k,\ell) \neq (k',\ell')$, $6^k 13^\ell \neq 6^{k'} 13^{\ell'}$. Therefore, we can solve the linear system for the unknown $c_{jk\ell}$'s and obtain the value of Holant$_\Omega$. □

The previous proof can be easily modified to reduce from #EO over 4-regular graphs by interpolating the so-called crossover signature. Conceptually, the current proof is simpler because the #P-hardness proof for #EO over 4-regular graphs in [31] reduces from the same starting point as our current proof.

It was shown that any symmetric signature with a rank 3 signature matrix defines a #P-hard Holant problem (see Corollary 5.7 [32]). The only nonplanar part of the proof is that the initial problem in the reduction, counting Eulerian orientations over 4-regular graphs, was not known to be #P-hard when the input is also input planar. The reductions themselves were all planar. Here we have shown the #P-hardness of this problem under the planarity restriction in Theorem 8, and therefore obtain the planar version of Corollary 5.7 in [32].

Corollary 9. *For a symmetric arity 4 signature* $[f_0, f_1, f_2, f_3, f_4]$ *with complex weights, if there does* not *exist* $a, b, c \in \mathbb{C}$*, not all zero, such that for all* $k \in \{0,1,2\}$*,* $af_k + bf_{k+1} + cf_{k+2} = 0$*, then* Pl-Holant$(f)$ *is* #P*-hard.*

With Corollary 9, only one obstacle remains in proving a dichotomy for a symmetric arity 4 signature in the Pl-Holant framework: the case $[v, 1, 0, 0, 0]$ when $v \neq 0$. We handle this by a reduction from Pl-Holant$([v, 1, 0, 0])$, which is #P-hard over planar graphs for $v \neq 0$. These problems are the weighted versions of counting matchings over planar k-regular graphs for $k = 4, 3$ respectively. This proof uses a refined interpolation technique. A planar $\mathcal{F}$-gate is essentially a planar graph gadget where the vertices are assigned signatures in $\mathcal{F}$.

Lemma 10. *Let* $\mathcal{F}$ *be a set of signatures. If there exists a planar* $\mathcal{F}$*-gate with signature matrix* $M \in \mathbb{C}^{2\times 2}$ *and a planar* $\mathcal{F}$*-gate with signature* $s \in \mathbb{C}^{2\times 1}$ *such that (1)* $\det(M) \neq 0$*, (2)* $\det([s\ Ms]) \neq 0$*, and (3)* M *has infinite order modulo a scalar, then* Pl-Holant$(\mathcal{F} \cup \{[a,b]\}) \leq_T$ Pl-Holant$(\mathcal{F})$ *for any* $a, b \in \mathbb{C}$*.*

The refinement is in the third condition. Previous work [35, 27, 29] used a stronger third condition: *the ratio of the eigenvalues of* M *is not a root of unity.* The first two conditions of Lemma 10 are easy to check. Our third condition holds in one of these two cases: either the eigenvalues are the same but M is not a multiple of the identity matrix, or the eigenvalues are different but their ratio is not a root of unity. The power of this lemma is that when our third condition fails to hold, we can construct M^{-1} by some constant number of copies of M and use this in other gadget constructions. This is called the anti-gadget technique [25]. We use this interpolation lemma or the anti-gadget technique to realize $[1,0,0]$.

To effectively use $[1,0,0]$, we introduce the notion of a planar pairing.

Definition 11 (Planar pairing). *A* planar pairing *in a graph $G=(V,E)$ is a set of edges $P \subset V \times V$ such that P is a perfect matching in the graph $(V, V \times V)$, and the graph $(V, E \cup P)$ is planar.*

Lemma 12. *For any planar 3-regular graph G, there exists a planar pairing that can be computed in polynomial time.*

With this lemma, we may use $[1,0,0]$ as $[1,0]$ on every vertex of a planar 3-regular graph, and obtain the hardness of the weighted versions of counting matchings over planar 4-regular graphs.

Lemma 13. *If $v \in \mathbb{C} - \{0\}$, then* Pl-Holant$([v,1,0,0,0])$ *is* #P-*hard.*

Combining Corollary 9 and Lemma 12 with Theorem 22 in [36], we can prove our Pl-Holant dichotomy for a symmetric arity 4 signature. A signature is vanishing if the Holant is always 0 [32].

Theorem 14. *If f is a non-degenerate, symmetric, complex-valued signature of arity 4 in Boolean variables, then* Pl-Holant(f) *is* #P-*hard unless f is $\mathscr{A}$-transformable or $\mathscr{P}$-transformable or vanishing or $\mathscr{M}$-transformable, in which case the problem is in* P.

4 Pinning for Planar Graphs

The idea of "pinning" is a common reduction technique between counting problems. For the #CSP framework, pinning fixes some variables to specific values of the domain by means of the constant functions [37, 30, 38, 31]. For counting graph homomorphisms, pinning is used when the input graph is connected and the target graph is disconnected. Pinning a vertex of the input graph to a vertex of the target graph forces all the vertices of the input graph to map to the same connected component of the target graph [39–42]. In the Boolean domain, the constant 0 and 1 functions are the signatures $[1,0]$ and $[0,1]$ respectively.

From these works, the most relevant pinning lemma for the Pl-#CSP framework is by Dyer, Goldberg, and Jerrum in [30], where they show how to pin in the #CSP framework. However, the proof of this pinning lemma is highly nonplanar. Cai, Lu, and Xia [23] overcame this difficultly in the proof of their dichotomy theorem for the real-weighted Pl-#CSP framework by first undergoing a holographic transformation by the Hadamard matrix $H = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ and then pinning in this Hadamard basis.[2] We stress that this holographic transformation is necessary. Indeed, if one were able to pin in the standard basis of the

[2] The pinning in [23], which is accomplished in Section IV, is not summarized in a single statement but is implied by the combination of all the results in that section.

Pl-#CSP framework, then P = #P would follow since Pl-#CSP($\widehat{\mathscr{M}}$) is tractable but Pl-#CSP($\widehat{\mathscr{M}} \cup \{[1,0],[0,1]\}$) is #P-hard by our main result, Theorem 19.

Since Pl-#CSP($\mathcal{F}$) is Turing equivalent to Pl-Holant($\mathcal{F} \cup \mathcal{EQ}$), its expression in the Hadamard basis is Pl-Holant($H\mathcal{F} \cup \widehat{\mathcal{EQ}}$). As $[1,0] \in \widehat{\mathcal{EQ}}$, pinning in this Hadamard basis amounts to obtaining the missing signature $[0,1]$.

Theorem 15 (Pinning). *Let $\mathcal{F}$ be any set of complex-weighted symmetric signatures. Then* Pl-Holant^c($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard (or in* P*) iff* Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard (or in* P*).*

This theorem does not exclude the possibility that either framework can express a problem of intermediate complexity. It merely says that if one framework does not contain a problem of intermediate complexity, then neither does the other. Our goal is to prove a dichotomy for Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$). By Theorem 15, this is equivalent to proving a dichotomy for Pl-Holant^c($\mathcal{F} \cup \widehat{\mathcal{EQ}}$).

In Theorem 15, the difference between the two counting problems is the presence of $[0,1]$ in the first problem. The proof is quite involved and can be found in the full version of this paper [1]. It is proved in several steps under various assumptions on $\mathcal{F}$. Each of these steps is proved in one of three ways:

1. either the first problem is tractable (so the second problem is as well);
2. or the second problem is #P-hard (so the first problem is as well);
3. or the first problem reduces to the second problem by constructing $[0,1]$ using the signatures from the second problem.

5 Main Dichotomy

Our main dichotomy theorem relies on a dichotomy for a single signature.

Theorem 16. *If f is a non-degenerate symmetric signature of arity at least 2 with complex weights in Boolean variables, then* Pl-Holant($\{f\} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard unless $f \in \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, in which case the problem is in* P.

We also prove a useful result that we call the Mixing Theorem.

Theorem 17 (Mixing). *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures. If $\mathcal{F} \subseteq \mathscr{A} \cup \widehat{\mathscr{P}} \cup \mathscr{M}$, then* Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard unless $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \widehat{\mathscr{P}}$, or $\mathcal{F} \subseteq \mathscr{M}$.*

By Theorems 15, 16, and 17, the proof of our main theorem is straightforward.

Theorem 18. *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-Holant($\mathcal{F} \cup \widehat{\mathcal{EQ}}$) *is* #P*-hard unless $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \widehat{\mathscr{P}}$, or $\mathcal{F} \subseteq \mathscr{M}$, in which case the problem is in* P.

We also have the corresponding theorem for the Pl-#CSP framework in the standard basis, which is equivalent to Theorem 1.

Theorem 19. *Let $\mathcal{F}$ be any set of symmetric, complex-valued signatures in Boolean variables. Then* Pl-#CSP($\mathcal{F}$) *is* #P*-hard unless $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{P}$, or $\mathcal{F} \subseteq \widehat{\mathscr{M}}$, in which case the problem is in* P.

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