Gadgets and Anti-Gadgets Leading to a Complexity Dichotomy

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Joint with:
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Michael Kowalczyk (Northern Michigan University)

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#VertexCover

Definition

A vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex in the set.

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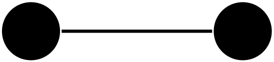

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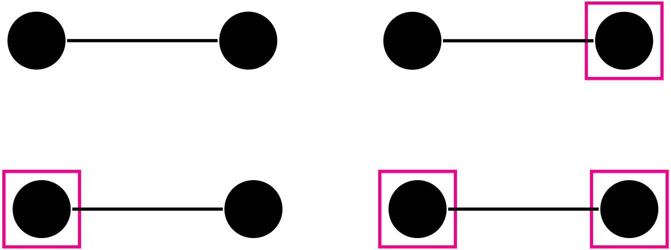

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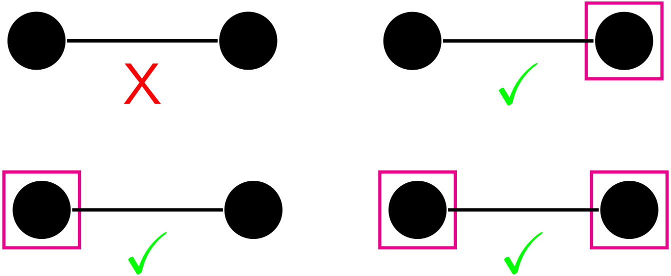
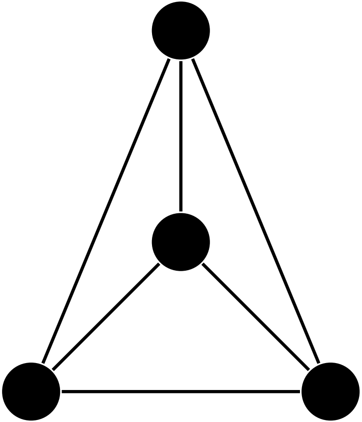


Systematic Approach to #VertexCover

- $G = (V, E)$



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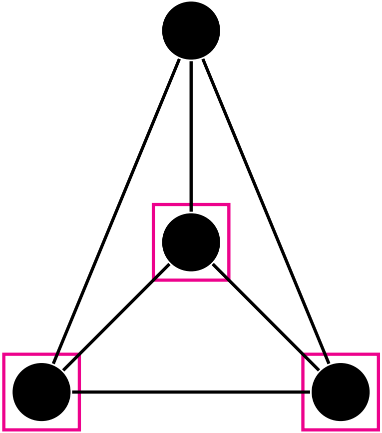

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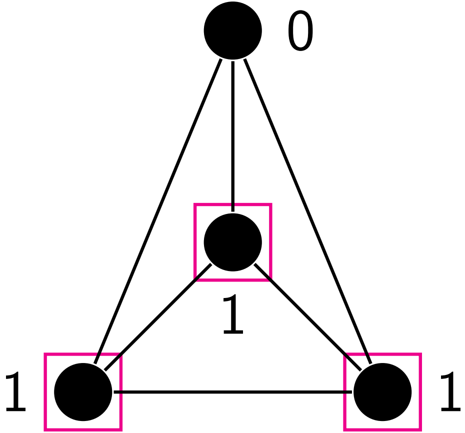
- $G = (V, E)$
- $\sigma : V \to \{0, 1\}$



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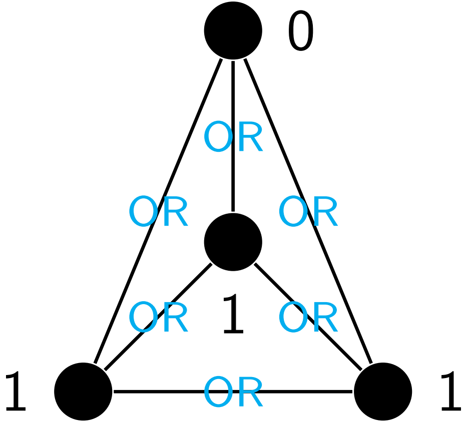

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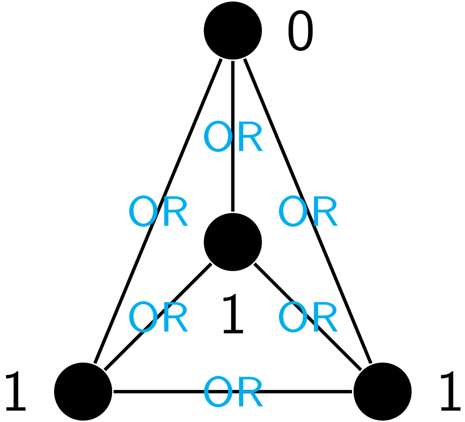

$$\prod_{(u,v)\in E} \mathrm{OR}(\sigma(u), \sigma(v)) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

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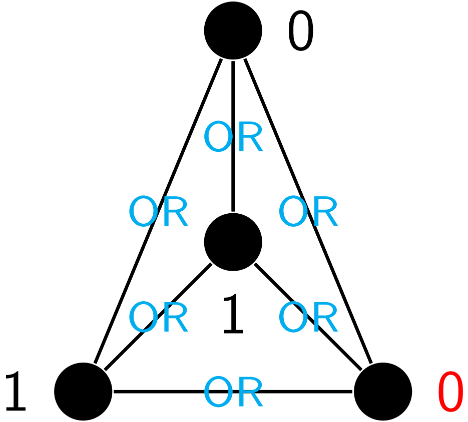

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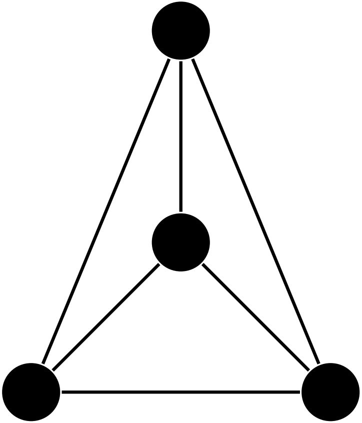

$$\#\textsc{VertexCover}(G) = \sum_{\sigma : V \to \{0,1\}} \prod_{(u,v) \in E} \mathrm{OR}(\sigma(u), \sigma(v))$$

Generalize

$$\sum_{\sigma: V \to \{0,1\}} \prod_{(u,v) \in E} \mathrm{OR}\left(\sigma(u), \sigma(v)\right)$$

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$$\sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E} f(\sigma(u),\sigma(v))$$

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p	q	$f(p,q)$
0	0	w
0	1	x
1	0	y
1	1	z

where $w, x, y, z \in \mathbb{C}$

Generalize

Partition Function: $Z(\cdot)$

$$Z(G) = \sum_{\sigma : V \to \{0,1\}} \prod_{(u,v) \in E} f(\sigma(u), \sigma(v))$$

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Main Result

Theorem (Dichotomy Theorem)

Over 3-regular graphs G, the counting problem for any (binary) complex-weighted function f

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is either computable in polynomial time or $\#\mathrm{P}$-hard. Furthermore, the complexity is efficiently decidable.

Outline

1. Related work
2. Define Holant function
3. Proof sketch
   - Anti-gadgets

Related Work: Dichotomy Theorems

- Symmetric f
  - $f(0,1) = f(1,0)$
- 3-regular graphs with outputs in
  - $\{0,1\}$ [Cai, Lu, Xia 08]
  - $\{0,1,-1\}$ [Kowalczyk 09]
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This work:

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- 3-regular graphs with outputs in
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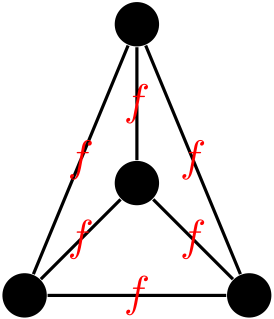
Definition of Holant Function

- Partition Function



$$\sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E} f\left(\sigma(u),\sigma(v)\right)$$

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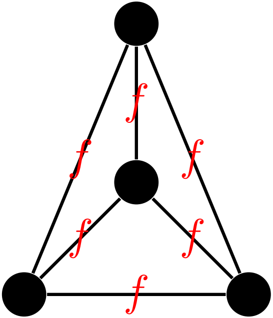
- Partition Function
  - Assignments to vertices
  - Functions on edges



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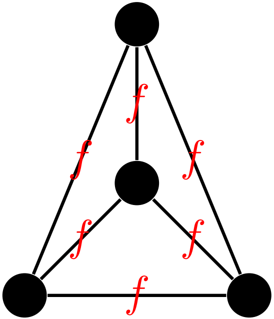

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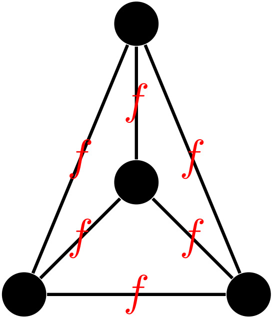

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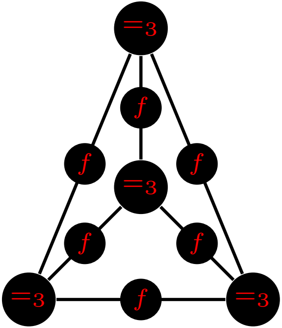

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Definition of Holant Function

- $\text{Holant}(\{{\color{red}f}\} \mid \{=_3\})$ is a counting problem defined over (2,3)-regular bipartite graphs.

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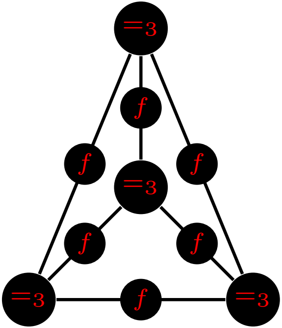

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- Degree 2 vertices take f.
- Degree 3 vertices take $=_3$.

- Holant Function
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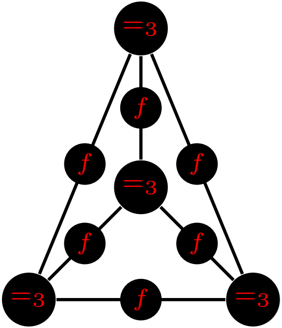

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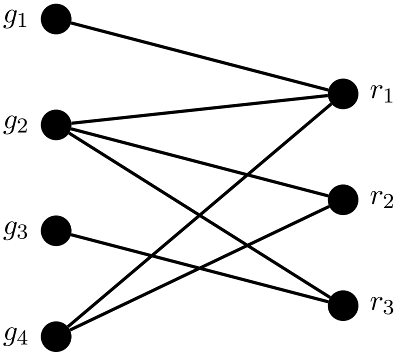
General Bipartite Holant Definition

- More generally, $\mathrm{Holant}(\mathcal{G} \,|\, \mathcal{R})$ is a counting problem defined over bipartite graphs.



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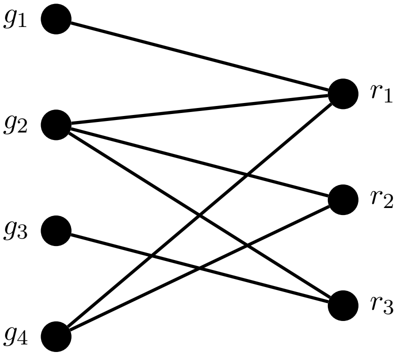

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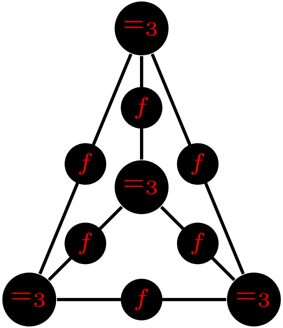
Symmetric vs Asymmetric Function



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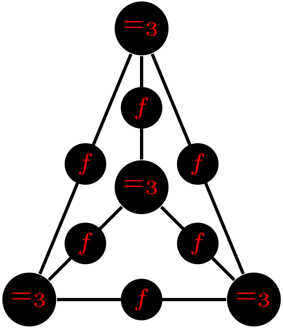

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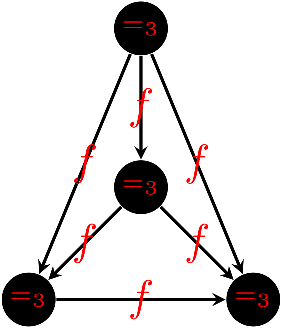

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Symmetric vs Asymmetric Function

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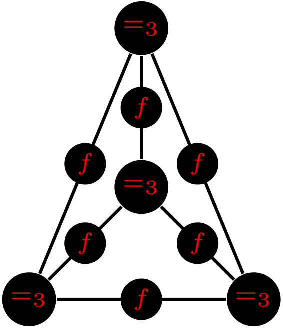

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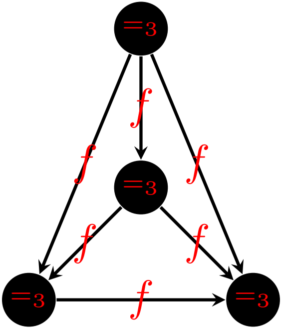
- Directed 3-regular



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Strategy for Proving #P-hardness

- $\#\textsc{VertexCover}$ is #P-hard over 3-regular graphs.
- $\operatorname{Holant}(\{\mathrm{OR}_2\} \mid \{=_3\})$ is $\#\textsc{VertexCover}$ on 3-regular graphs.

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where $\mathcal{U}$ is the set of all unary functions.

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- Obtain $\mathcal{U}$ via interpolation.

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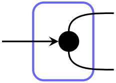
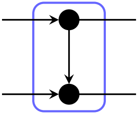
- Distinct evaluation points $\iff$ unary functions pairwise linearly independent (as length-2 vectors).

Construction of Unary Functions

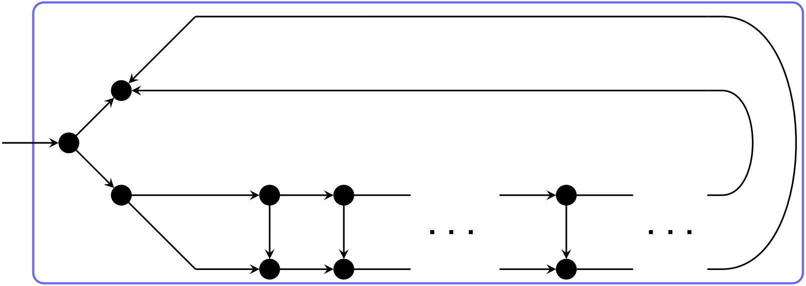
Projective Gadget



Recursive Gadget

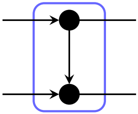


Unary Function



Matrix Representation

- Left side indexes the row.
- Right side indexes the column.
- High order bit on top.



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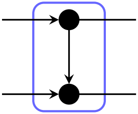

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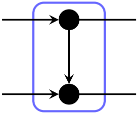

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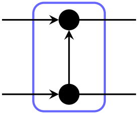

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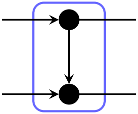

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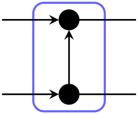

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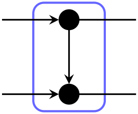
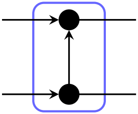


$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$$



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- Matrix of the composition is the product of the component matrices.

Anti-Gadget Construction

- Want set of matrix powers to form an infinite set of pairwise linearly independent matrices.

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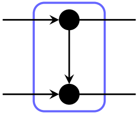

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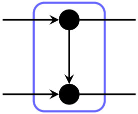

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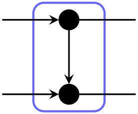

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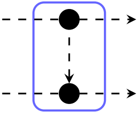

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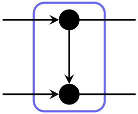

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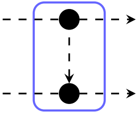

$$\left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \right)^{-1}$$

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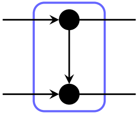

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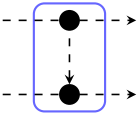

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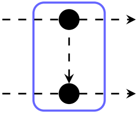
Anti-Gadget Technique



$$\left(\begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix}\right)^{-1} \left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2}\right)^{-1}$$

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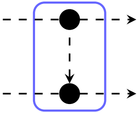

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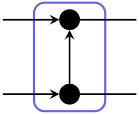

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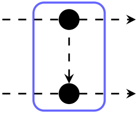
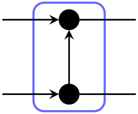


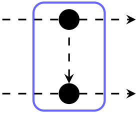
$$\left(\begin{bmatrix} w & 0 & 0 & 0 \\ 0 & {\color{red}x} & 0 & 0 \\ 0 & 0 & {\color{red}y} & 0 \\ 0 & 0 & 0 & z \end{bmatrix}\right)^{-1} \left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2}\right)^{-1}$$



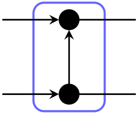
$$\begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & {\color{red}y} & 0 & 0 \\ 0 & 0 & {\color{red}x} & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$$

- The composition of these two gadgets yields...

Anti-Gadget Technique



$$\left(\begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix}\right)^{-1} \quad \left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2}\right)^{-1}$$



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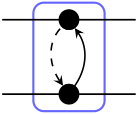

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{y}{x} & 0 & 0 \\ 0 & 0 & \frac{x}{y} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The First Anti-Gadget Lemma

Lemma

For $w, x, y, z \in \mathbb{C}$, *if*

- $wz \neq xy$,
- $wxyz \neq 0$, *and*
- $|x| \neq |y|$,

then there exists a recursive gadget whose matrix powers form an infinite set of pairwise linearly independent matrices.

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then there exists a recursive gadget whose matrix powers form an infinite set of pairwise linearly independent matrices.

Corollary

For $w, x, y, z \in \mathbb{C}$ *as above,* $\text{Holant}(\{f\} \mid \{=_3\})$ *is* $\#\text{P}$*-hard.*

Thank You

Thank You

Paper and slides available on my website.
www.cs.wisc.edu/~tdw