Holant, Dichotomy Theorems, and Interpolation

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Preliminary Examination
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Outline

1. **Introduction**
2. **Previous Work**
   - Dichotomy for $Z_3(\vec{G}; f)$
   - Dichotomy for PI-#CSP($\mathcal{F}$)
   - Dichotomy for Holant($\mathcal{F}$)
3. **Current Work**
4. **Future Work**

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- Polynomial Interpolation
  - Main reduction technique for proving hardness.

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- Holographic Transformation
  - Change of basis

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#VertexCover

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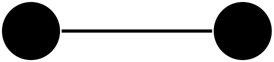

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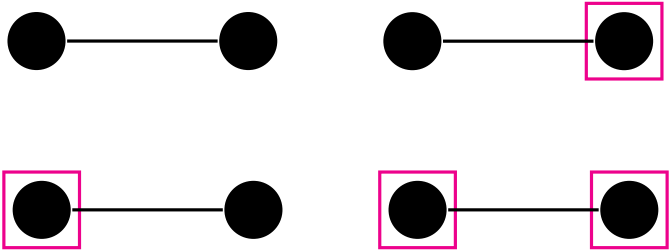

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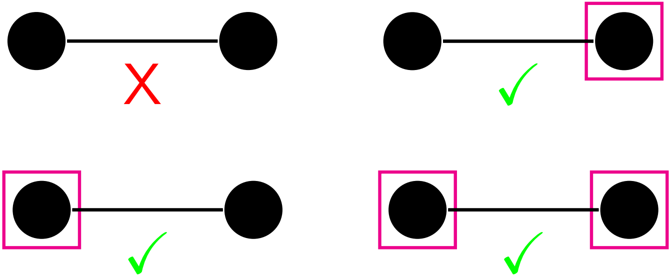
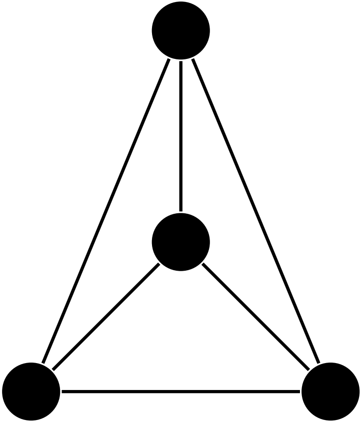


Systematic Approach to #VertexCover

- $G = (V, E)$



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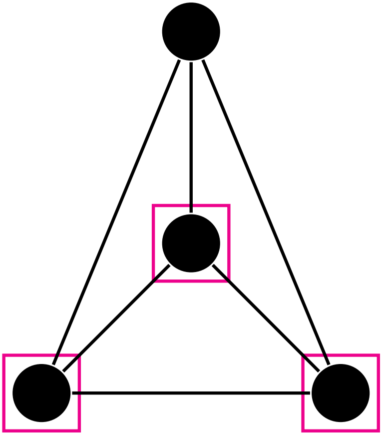

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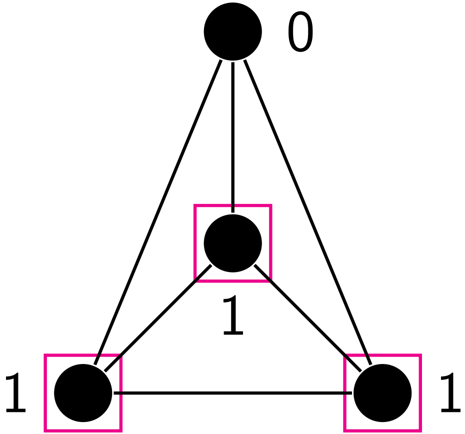
- $G = (V, E)$
- $\sigma : V \to \{0, 1\}$



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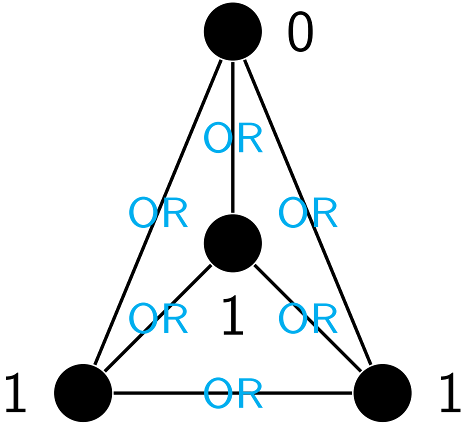

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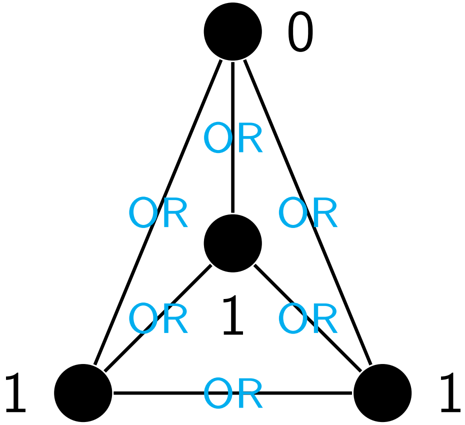

$$\prod_{(u,v)\in E} \mathrm{OR}(\sigma(u), \sigma(v)) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

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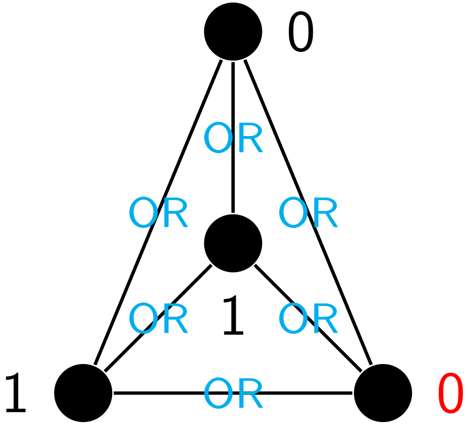

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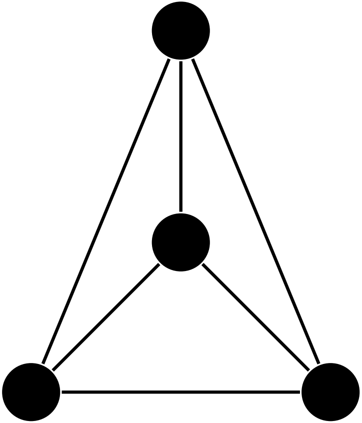
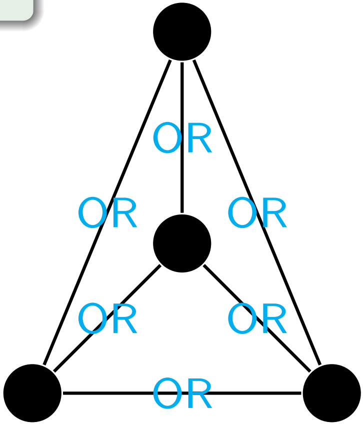


$$\#\text{VertexCover}(G) = \sum_{\sigma: V \to \{0,1\}} \prod_{(u,v) \in E} \text{OR}(\sigma(u), \sigma(v))$$

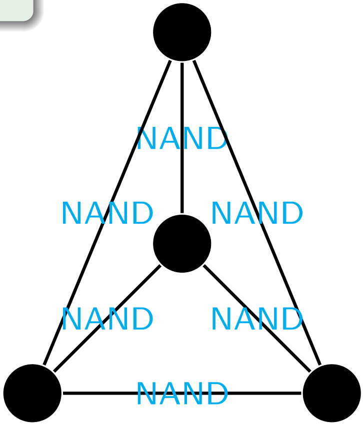
Example

- OR corresponds to #VertexCover



Example

- OR corresponds to #VertexCover
- NAND corresponds to #IndependentSet



Other Edge Constraints

Example

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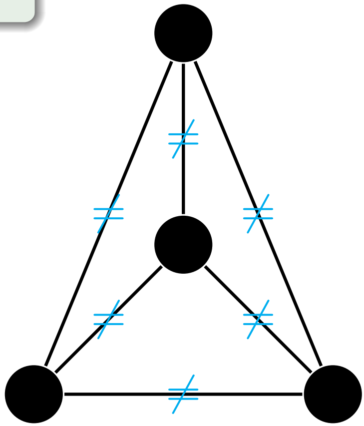
- $\neq$ corresponds to #Bipartition



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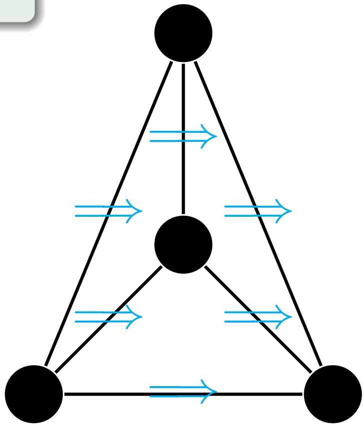

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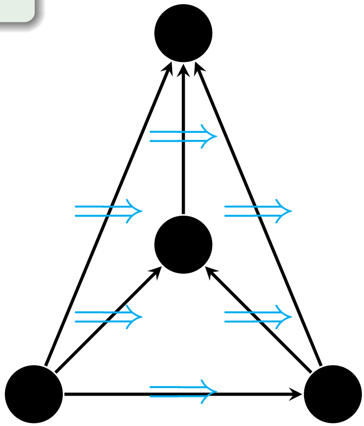

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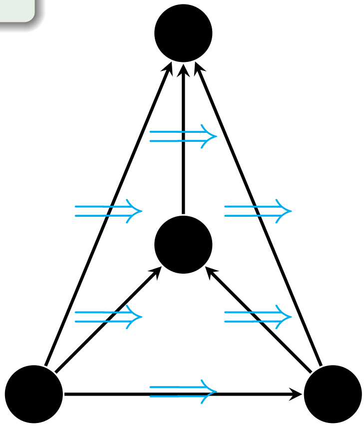

Generalize

$$\sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E} \mathrm{OR}(\sigma(u),\sigma(v))$$

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p	q	$\mathrm{OR}(p, q)$
0	0	0
0	1	1
1	0	1
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$$\sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E} f(\sigma(u),\sigma(v))$$

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Input		Output
p	q	$f(p,q)$
0	0	w
0	1	x
1	0	y
1	1	z

where $w, x, y, z \in \mathbb{C}$

Partition Function:

$$Z(\vec{G}; f) = \sum_{\sigma : V \to \{0,1\}} \prod_{(u,v) \in E} f(\sigma(u), \sigma(v))$$

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Theorem (Cai, Kowalczyk, W 12)

For 3-regular $\vec{G}$,

$$Z(\vec{G}; f) = \sum_{\sigma: V \to \{0,1\}} \prod_{(u,v) \in E} f(\sigma(u), \sigma(v))$$

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Explicit form for tractable cases.

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where $\mathcal{U}$ is the set of all unary signatures.

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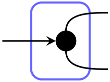
- Distinct evaluation points $\Leftrightarrow$ $(g_i(0), g_i(1))$ pairwise linearly independent

Construction of Unary Signatures

Projective Gadget



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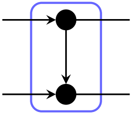
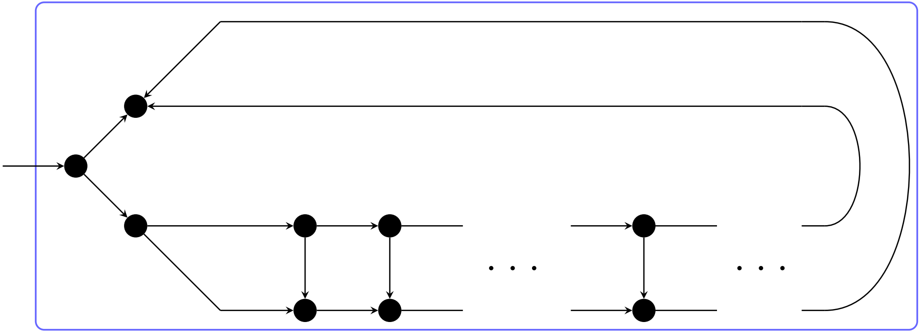


Unary Signature



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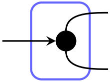

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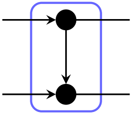

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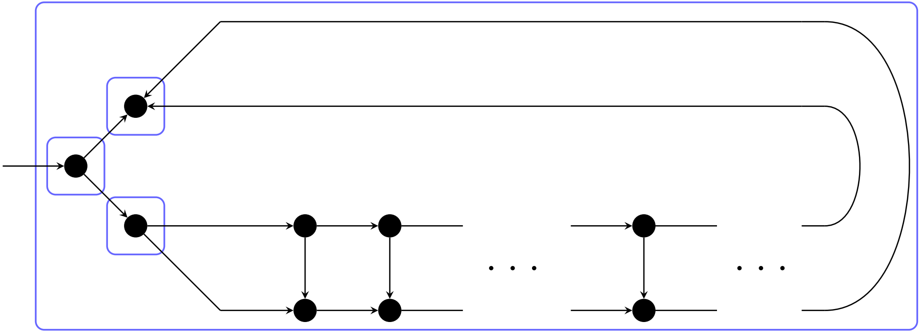

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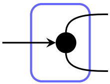

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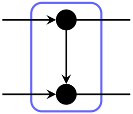

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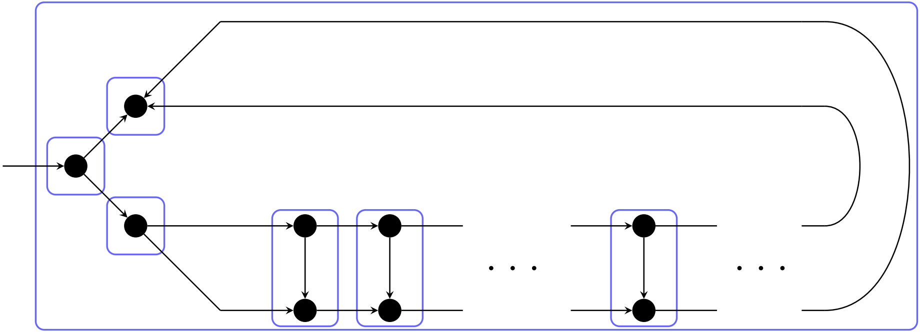

Definition

Weighted truth table for a signature $g(a, b, c, d) = g^{abcd}$ written as

$$\mathrm{SM}(g) = \begin{bmatrix} g^{0000} & g^{0010} & g^{0001} & g^{0011} \\ g^{0100} & g^{0110} & g^{0101} & g^{0111} \\ g^{1000} & g^{1010} & g^{1001} & g^{1011} \\ g^{1100} & g^{1110} & g^{1101} & g^{1111} \end{bmatrix}$$

is called its signature matrix.

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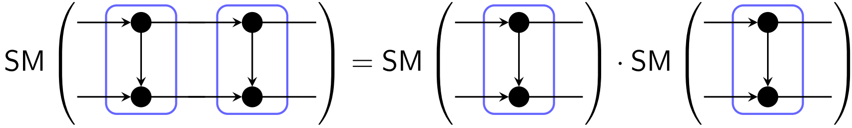
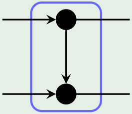
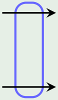
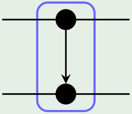


Example

SM () = SM () · SM ()

$$= \begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$$

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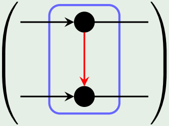
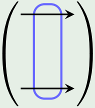
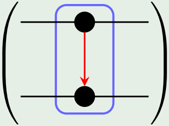
SM  = SM  · SM 

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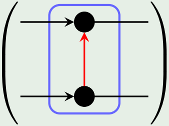
SM 

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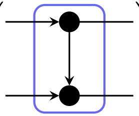
SM () $= \begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$

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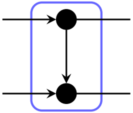
SM () $= \begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$

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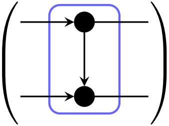
SM  $= \begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$

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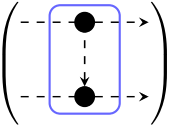
SM  $=$

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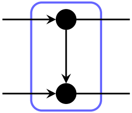
SM ()

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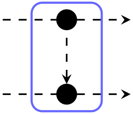
SM ()

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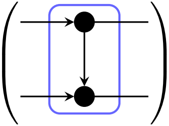
SM  $= \begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$

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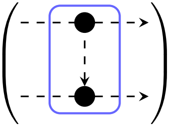
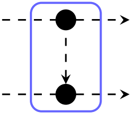
SM  $= \left(\begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \right)^{-1}$

SM ()

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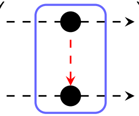
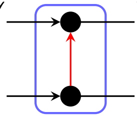
Anti-Gadget Technique

SM () $= \left(\begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \right)^{-1}$

SM () $= \begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$

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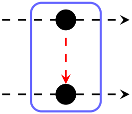
SM () =

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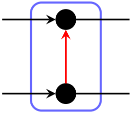
SM () =

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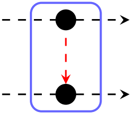
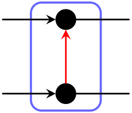
Composition of these two gadgets yields...

SM () $= \left(\begin{bmatrix} w & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix}\right)^{-1} \left(\begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2}\right)^{-1}$

SM () $= \begin{bmatrix} w & x \\ y & z \end{bmatrix}^{\otimes 2} \begin{bmatrix} w & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$

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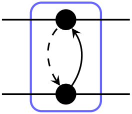
SM () $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{y}{x} & 0 & 0 \\ 0 & 0 & \frac{x}{y} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

First Lemma Using Anti-Gadgets

Lemma

For $w, x, y, z \in \mathbb{C}$, *if*

- $wz \neq xy$,
- $wxyz \neq 0$, *and*
- $|x| \neq |y|$,

then there exists a recursive gadget whose matrix powers form an infinite set of pairwise linearly independent matrices.

Lemma

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then there exists a recursive gadget whose matrix powers form an infinite set of pairwise linearly independent matrices.

Corollary

For $w, x, y, z \in \mathbb{C}$ *as above,* $\text{Holant}(f \mid =_3)$ *is* $\#\mathrm{P}$*-hard.*

Outline

1. Introduction
2. **Previous Work**
   - Dichotomy for $Z_3(\vec{G}; f)$
   - Dichotomy for PI-#CSP($\mathcal{F}$)
   - Dichotomy for Holant($\mathcal{F}$)
3. Current Work
4. Future Work

Constraint Graph for $\#\text{CSP}(\mathcal{F})$ Instance

$\mathcal{F} = \{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_3\}$

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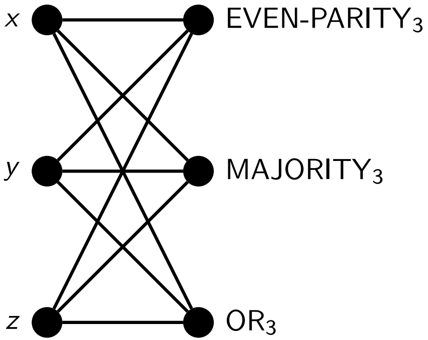

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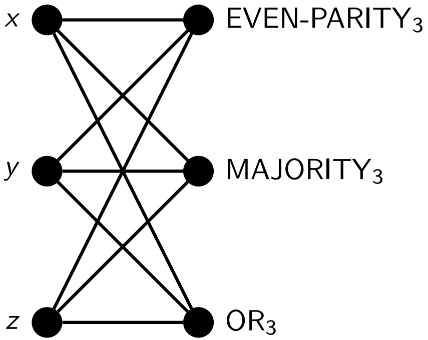

NOT planar, so **NOT** an instance of PI-$\#\text{CSP}(\{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_3\})$

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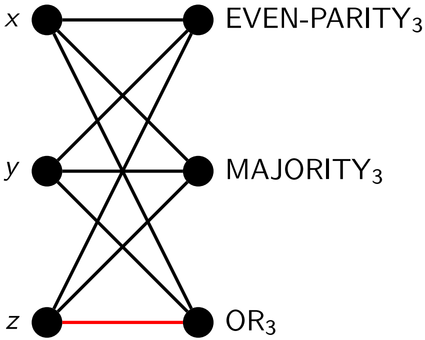

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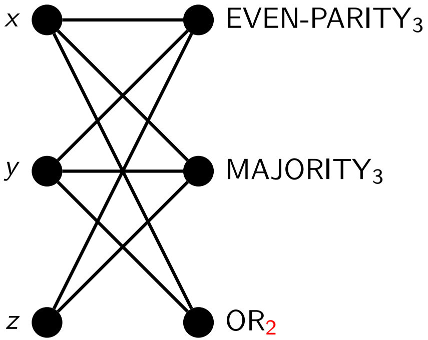

Constraint Graph for $\#\mathrm{CSP}(\mathcal{F})$ Instance

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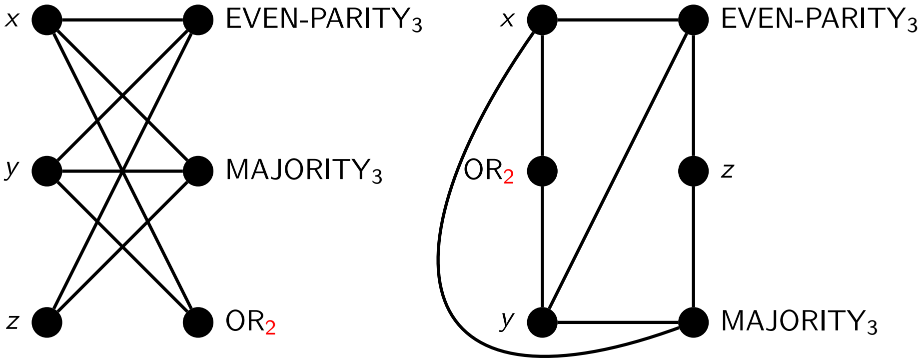

VALID instance of PI-$\#\mathrm{CSP}(\{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_2\})$

$\#\mathrm{CSP}(\mathcal{F})$

- On input with (bipartite) constraint graph $G = (V, C, E)$, compute

$$\sum_{\sigma: V \to \{0,1\}} \prod_{c \in C} f_c \left(\sigma \mid_{N(c)} \right),$$

where $N(c)$ are the neighbors of c.

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Holant($\mathcal{F}$)

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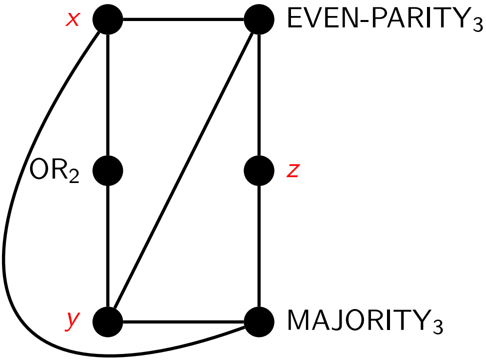
$$\sum_{\sigma: E \to \{0,1\}} \prod_{v \in V} f_v \left(\sigma \mid_{E(v)} \right),$$

where $E(v)$ are the incident edges of v.

$\#\mathrm{CSP}(\mathcal{F})$ in Holant Framework



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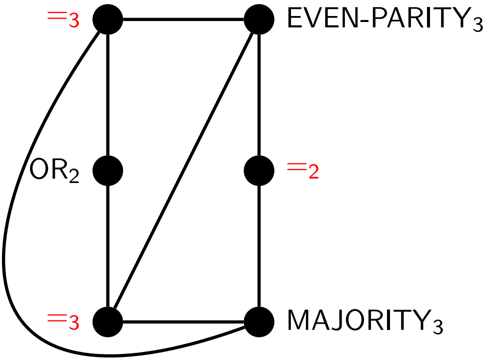
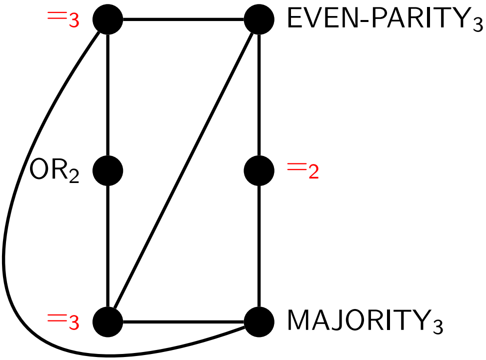
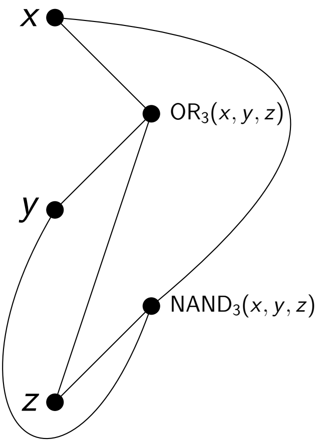




$$\#\text{CSP}(\mathcal{F}) \equiv_T \text{Holant}(\mathcal{EQ} \mid \mathcal{F}),$$

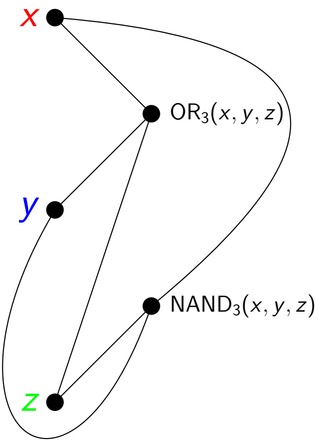
where $\mathcal{EQ} = \{=_1, =_2, =_3, \dots\}$ is the set of equalities of all arities.

Visualizing a Holographic Transformation



Visualizing a Holographic Transformation

$(1\ 0\ 0\ 1)_x \quad (1\ 0\ 0\ 1)_y \quad (1\ 0\ 0\ 1)_z$



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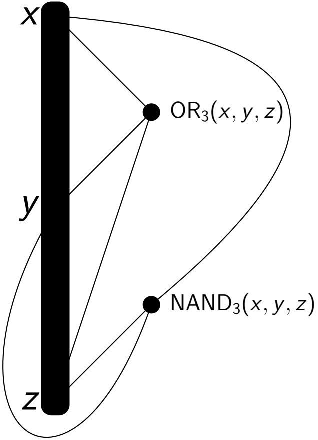

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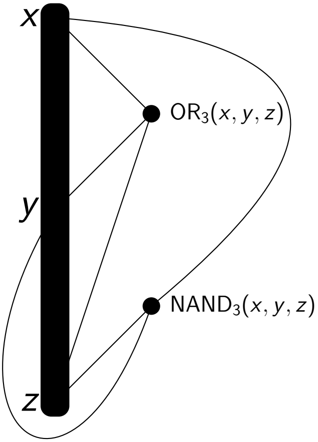

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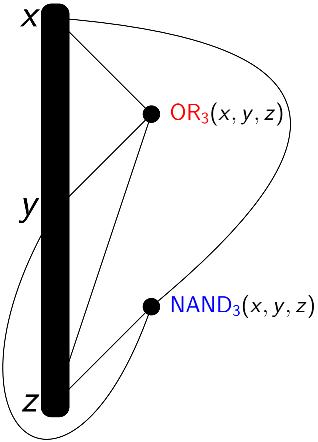

$$\begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}_{\mathrm{OR}_3}$$

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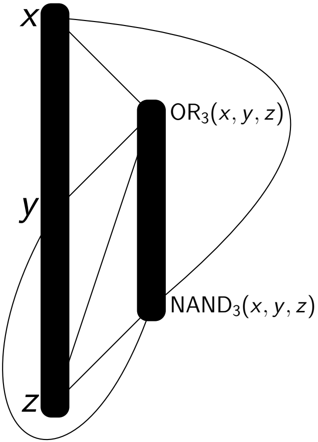

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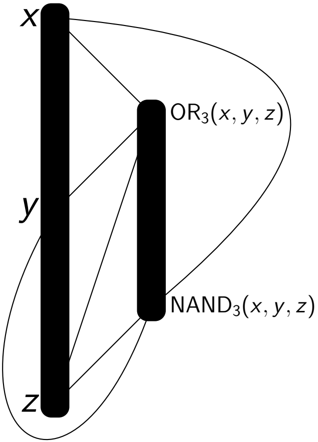

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{\text{OR}_3} \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}_{\text{NAND}_3}$$

Visualizing a Holographic Transformation

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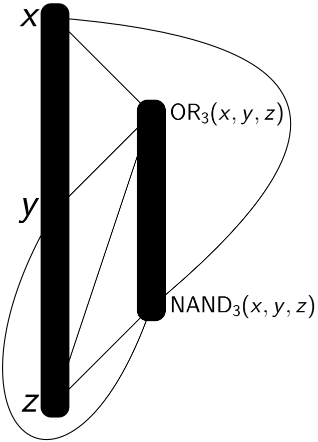
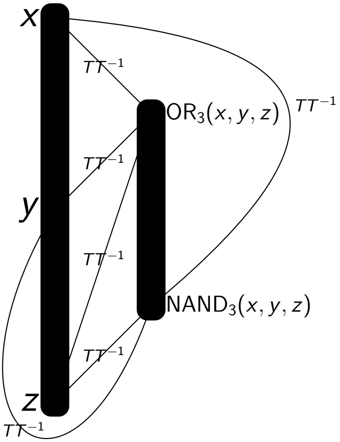


Visualizing a Holographic Transformation

$$(1\ 0\ 0\ 1)_x \otimes (1\ 0\ 0\ 1)_y \otimes (1\ 0\ 0\ 1)_z\, T^{\otimes 6} (T^{-1})^{\otimes 6}$$



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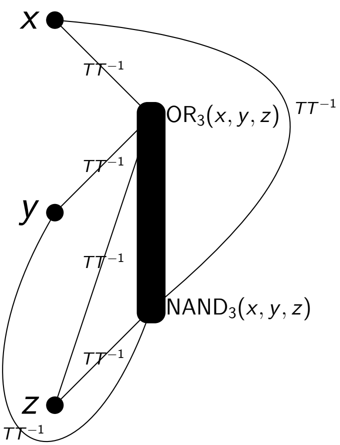

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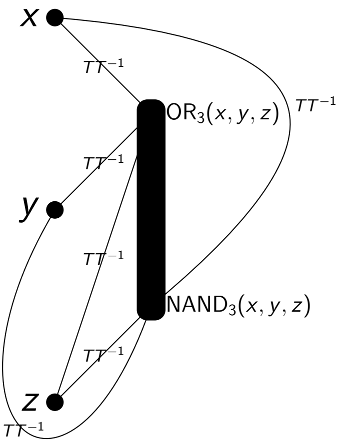
Visualizing a Holographic Transformation

$$(1\ 0\ 0\ 1)_x \otimes (1\ 0\ 0\ 1)_y \otimes (1\ 0\ 0\ 1)_z \left(T^{\otimes 2}\right)^{\otimes 3} (T^{-1})^{\otimes 6} \begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}_{\mathrm{OR}_3} \otimes \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\0\end{pmatrix}_{\mathrm{NAND}_3}$$



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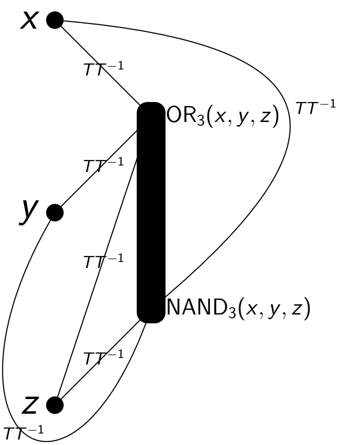

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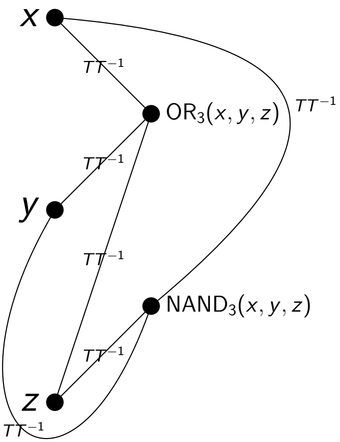

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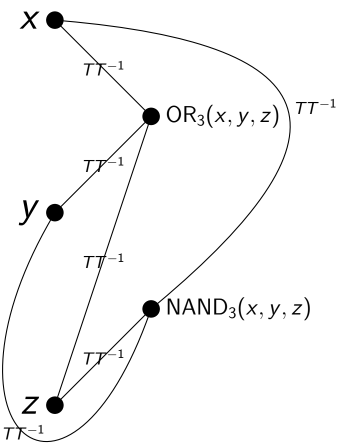

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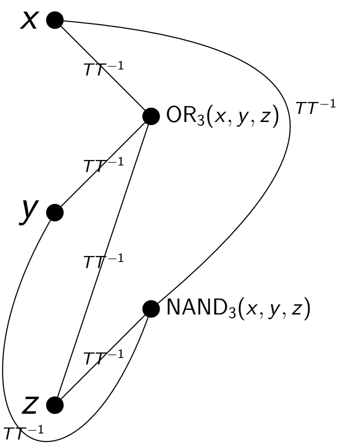
$$(1\ 0\ 0\ 1)_x T^{\otimes 2} \otimes (1\ 0\ 0\ 1)_y T^{\otimes 2} \otimes (1\ 0\ 0\ 1)_z T^{\otimes 2}$$



$$(T^{-1})^{\otimes 3} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{OR_3} \otimes (T^{-1})^{\otimes 3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}_{NAND_3}$$

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$$\mathrm{AND}_3 = [0, 0, 0, 1]$$

$$\text{EVEN-PARITY}_4 = [1, 0, 1, 0, 1]$$

$$\mathrm{MAJORITY}_5 = [0, 0, 0, 1, 1, 1]$$

$$(=_6) = \mathrm{EQUALITY}_6 = [1, 0, 0, 0, 0, 0, 1]$$

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$$\begin{aligned}
(=_n)H^{\otimes n} &= \left\{ \begin{pmatrix} 1 & 0 \end{pmatrix}^{\otimes n} + \begin{pmatrix} 0 & 1 \end{pmatrix}^{\otimes n} \right\} H^{\otimes n} \\
&= \left\{ \begin{pmatrix} 1 & 0 \end{pmatrix} H \right\}^{\otimes n} + \left\{ \begin{pmatrix} 0 & 1 \end{pmatrix} H \right\}^{\otimes n} && \text{(mixed-product property)} \\
&= \begin{pmatrix} 1 & 1 \end{pmatrix}^{\otimes n} + \begin{pmatrix} 1 & -1 \end{pmatrix}^{\otimes n} \\
&= [2, 0, 2, 0, 2, 0, 2, \dots] && (n + 1 \text{ entries}) \\
&= 2 \cdot \text{EVEN-PARITY}_n
\end{aligned}$$

Some Signature Sets

Affine signatures $\mathscr{A}$:

1. $[1, 0, \ldots, 0, \pm 1]$
2. $[1, 0, \ldots, 0, \pm i]$
3. $[1, 0, 1, 0, \ldots, 0 \text{ or } 1]$
4. $[1, -i, 1, -i, \ldots, (-i) \text{ or } 1]$
5. $[0, 1, 0, 1, \ldots, 0 \text{ or } 1]$
6. $[1, i, 1, i, \ldots, i \text{ or } 1]$
7. $[1, 0, -1, 0, 1, 0, -1, 0, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$
8. $[1, 1, -1, -1, 1, 1, -1, -1, \ldots, 1 \text{ or } (-1)]$
9. $[0, 1, 0, -1, 0, 1, 0, -1, \ldots, 0 \text{ or } 1 \text{ or } (-1)]$
10. $[1, -1, -1, 1, 1, -1, -1, 1, \ldots, 1 \text{ or } (-1)]$

Product-type signatures $\mathscr{P}$:

1. $[0, x, 0]$
2. $[y, 0, \ldots, 0, z]$ (includes all unary signatures)

Some Signature Sets

Matchgate signatures $\mathscr{M}$:

1. $[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n]$
2. $[\alpha^n, 0, \alpha^{n-1}\beta, 0, \ldots, 0, \alpha\beta^{n-1}, 0, \beta^n, 0]$
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They satisfy

- Parity condition
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They satisfy

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Example

$$\mathcal{EQ}H = \{2 \cdot \text{EVEN-PARITY}_n \mid n \in \mathbb{Z}^+\}$$

Dichotomy Theorem

Theorem (Guo, W 13)

PI-#CSP($\mathcal{F}$) *is* #P*-hard unless* $\mathcal{F} \subseteq \mathscr{A}$, $\mathcal{F} \subseteq \mathscr{P}$, *or* $\mathcal{F} \subseteq H\mathscr{M}$,
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&\leq_T \text{PI-Holant}(\mathscr{M})
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is tractable by reduction to counting perfect matchings in planar graphs.

Relation to Previous Work: Planar Dichotomy Theorems

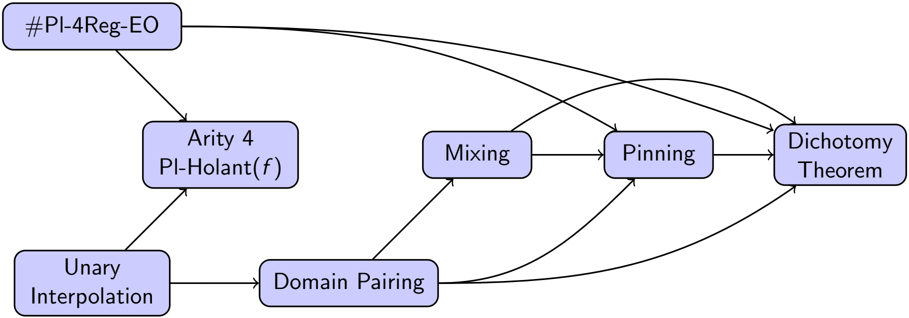
[Cai, Lu, Xia 10]

- Pl-#CSP($\mathcal{F}$) with real weights
- Pl-Holant([a, b, c, d]) with complex weights

[Cai, Kowalczyk 10]

- Pl-#CSP([a, b, c]) with complex weights

Proof Outline: Dependency Graph



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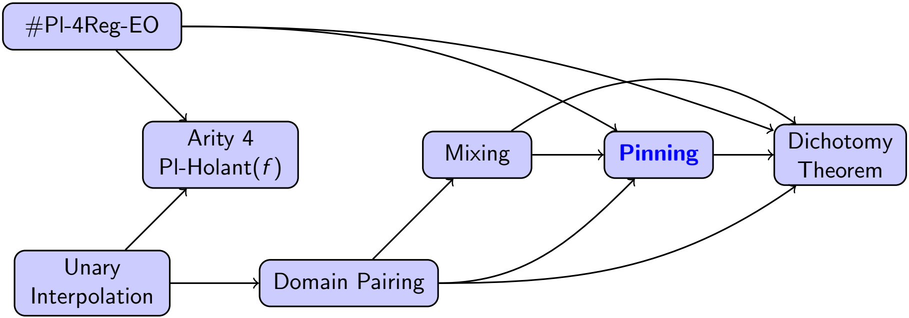

Pinning

Graph Homomorphism

- [Dyer, Greenhill 00]
- [Bulatov, Grohe 05]
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For any set of signatures $\mathcal{F}$ *with real weights,*

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Outline

1. Introduction
2. **Previous Work**
   - Dichotomy for $Z_3(\vec{G}; f)$
   - Dichotomy for PI-#CSP($\mathcal{F}$)
   - Dichotomy for Holant($\mathcal{F}$)
3. Current Work
4. Future Work

Definition

A signature grid $\Omega = (G, \mathcal{F})$ consists of

- a graph $G = (V, E)$,
- a set of signatures $\mathcal{F}$ with $\{0, 1\}$ inputs and a $\mathbb{C}$ output, and
- f_v is the signature on vertex v.

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Holant Framework

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- $f_v = [3, 0, 1, 0, 3]$ gives #4-Reg-EulerianOrientation

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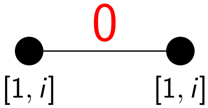

$1 \cdot 1$

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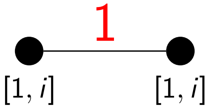

$1 \cdot 1 + i \cdot i$

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$$1 \cdot 1 + i \cdot i = 0$$

Single Signature Dichotomy

Theorem (Cai, Guo, W 13)

Holant(f) *is* #P*-hard unless*

1. f *is degenerate,*
2. f *is binary,*
3. f *is* $\mathscr{A}$*-transformable,*
4. f *is* $\mathscr{P}$*-transformable, or*
5. f *is vanishing,*

which are computable in polynomial time.

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Holant($\mathcal{F}$) *is* $\#\mathrm{P}$*-hard unless*

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4. $\mathcal{F} \subseteq \{$*vanishing*$\} \cup \{$*special binary*$\}$*, or*
5. $\mathcal{F} \subseteq \{$ *"highly" vanishing*$\} \cup \{$*special binary*$\} \cup \{$*degenerate*$\}$,

which are computable in polynomial time.

Single signature:

- Holant($[a, b, c, d]$) with complex weights [Cai, Huang, Lu 10]
- Holant($[a, b, c] \mid =_k$) with complex weights [Cai, Kowalczyk 11]

Signature set:

- Holant$^*(\mathcal{F})$ with complex weights [Cai, Lu, Xia 09]
- Holant$^c(\mathcal{F})$ with complex weights [Cai, Huang, Lu 10]
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- Holant($\mathcal{F}$) with real weights [Huang, Lu 12]

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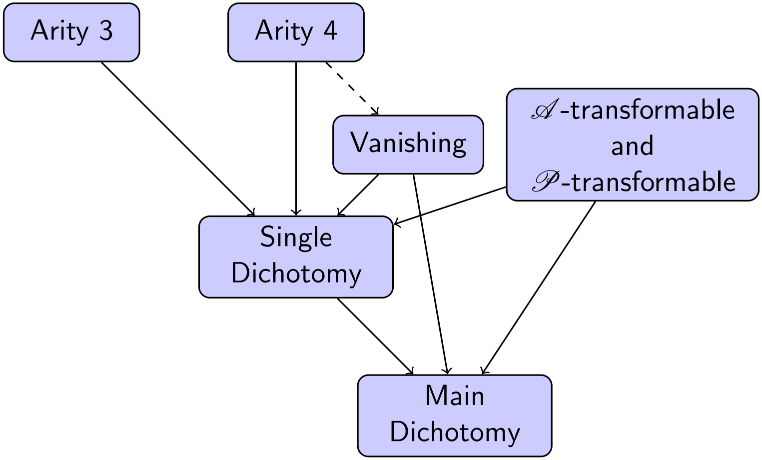

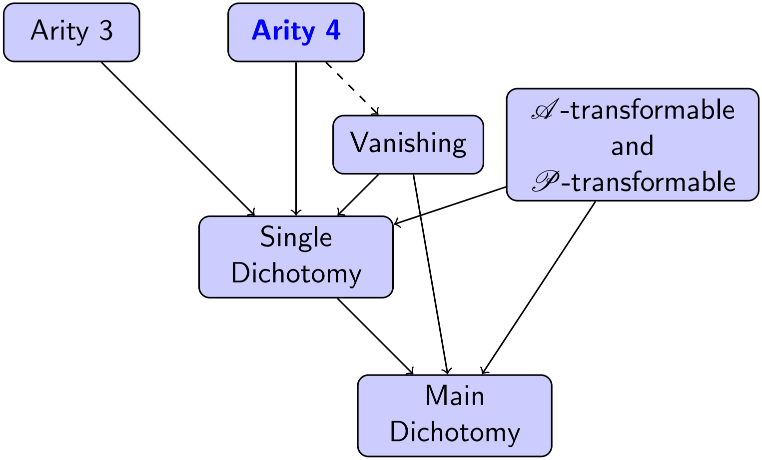
Arity 3
Arity 4
Vanishing
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and
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Single
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Main
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Redundant Signature Matrix

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- identical middle two rows and
- identical middle two columns.

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Example

$$\mathrm{SM}([f_0, f_1, f_2, f_3, f_4]) = \begin{bmatrix} f_0 & f_1 & f_1 & f_2 \\ f_1 & f_2 & f_2 & f_3 \\ f_1 & f_2 & f_2 & f_3 \\ f_2 & f_3 & f_3 & f_4 \end{bmatrix}$$

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Let $\mathrm{SM}(f) = M_f$.

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Let $\mathrm{RM}_4(\mathbb{C})$ be the set of 4-by-4 redundant matrices.

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$$\varphi : \mathrm{RM}_4(\mathbb{C}) \to \mathbb{C}^{3\times 3}$$

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Let $\varphi(M) = \widetilde{M}$ and $\psi = \varphi^{-1}$.

Identity of $\mathrm{RM}_4(\mathbb{C})$

Let g have signature matrix

$$M_g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\widetilde{M_g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Lemma (Cai, Guo, W 13)

Holant(g)

Let g have signature matrix

$$M_g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then

$$\widetilde{M_g} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Lemma (Cai, Guo, W 13)

$$\operatorname{Holant}([3,0,1,0,3]) \leq_T \operatorname{Holant}(g)$$

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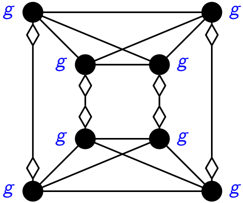
Consider an instance Ω of PI-Holant(g) with n vertices.



Interpolation

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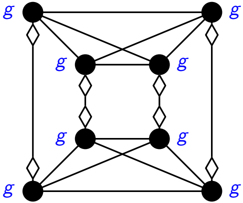
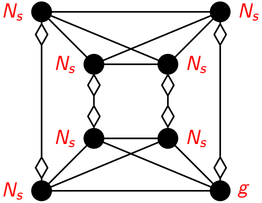
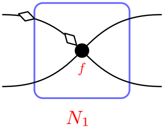
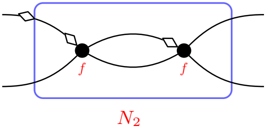
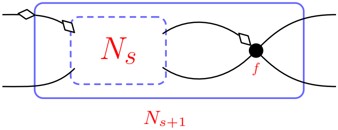


Construct instance Ω_s of PI-Holant(f) using N_s









Interpolation

By the Jordan normal form of $\widetilde{M_f}$, there exists $T, \Lambda \in \mathbb{C}^{3\times 3}$ such that

$$\widetilde{M_f} = T\Lambda T^{-1} = T \begin{bmatrix} \lambda_1 & b_1 & 0 \\ 0 & \lambda_2 & b_2 \\ 0 & 0 & \lambda_3 \end{bmatrix} T^{-1},$$

where $b_1, b_2 \in \{0, 1\}$.

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Notice

$$\widetilde{M_g} = T\widetilde{M_g} T^{-1}.$$

Interpolation

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   (Holant unchanged)
2. Then replace M_g with $\psi(\Lambda^s)$.

Stratify

We stratify all assignments to M_g in Ω' according to:

- $(0, 0)$ or $(2, 2)$ i many times;
- $(1, 1)$ j many times;
- $(0, 1)$ k many times;
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All other assignments contribute a factor 0.

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Let $c_{ijk\ell m}$ be the sum over all such assignments of the products of evaluations from $\psi(T)$ and $\psi(T^{-1})$ but excluding M_g on Ω'.

$$\text{Holant}_{\Omega} = \sum_{i+j=n} \tfrac{c_{ij000}}{2^j}.$$

The value of the Holant on Ω_s, for $s \geq 1$, is

$$\begin{aligned}\text{Holant}_{\Omega_s} &= \sum_{i+j+k+\ell+m=n} \lambda^{(i+j)s} \left(s\lambda^{s-1}\right)^{k+\ell} \left(s(s-1)\lambda^{s-2}\right)^{m} \left(\frac{c_{ijk\ell m}}{2^{j+k+m}}\right) \\ &= \lambda^{ns} \sum_{i+j+k+\ell+m=n} s^{k+\ell+m}(s-1)^{m} \left(\frac{c_{ijk\ell m}}{\lambda^{k+\ell+2m}2^{j+k+m}}\right).\end{aligned}$$

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In the linear system,

- rows are indexed by s and
- columns are indexed by (i, j, k, ℓ, m).

Rank Deficient

The linear system is rank deficient. Define new unknowns for any

$$0 \le q, m \qquad \text{and} \qquad q + m \le n,$$

$$x_{q,m} = \sum_{\substack{k+\ell=q \\ i+j=n-q-m}} \left(\frac{c_{ijk\ell m}}{\lambda^{k+\ell+2m} 2^{j+k+m}} \right).$$

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Let $\alpha_{q,m} = s^{q+m}(s-1)^m$.

Rank Deficient Again

New system still rank deficient since

$$s^{q+m}(s-1)^m = s^{q-1+m}(s-1)^m + s^{q-2+m+1}(s-1)^{m+1}.$$

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Thus,

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We recursively define new variables

$$x_{q-1,m} \leftarrow x_{q,m} + x_{q-1,m}$$

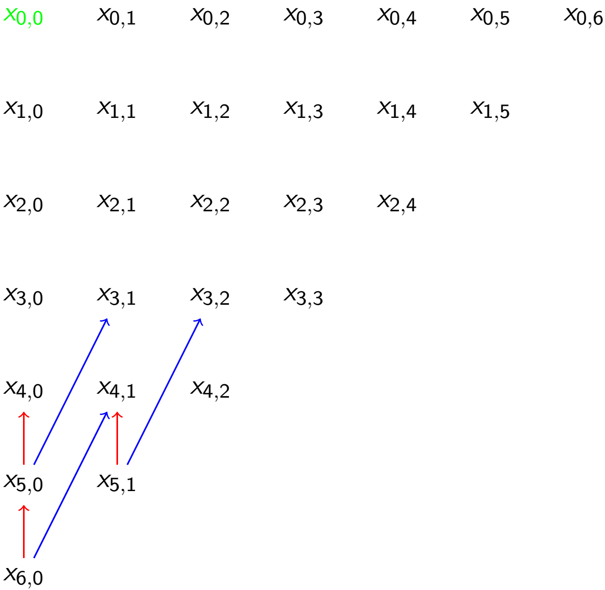
$$x_{q-2,m+1} \leftarrow x_{q,m} + x_{q-2,m+1}$$

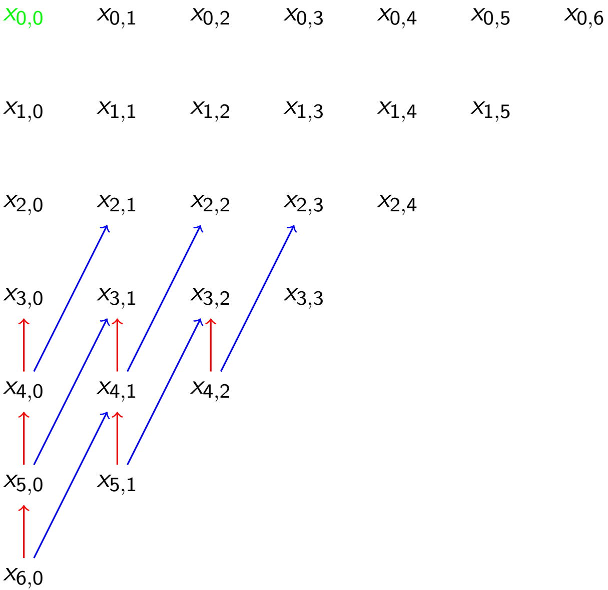
from $q = n$ down to 2.

$$
\begin{array}{ccccccc}
x_{0,0} & x_{0,1} & x_{0,2} & \cdots & x_{0,n-2} & x_{0,n-1} & x_{0,n} \\
x_{1,0} & x_{1,1} & x_{1,2} & \cdots & x_{1,n-2} & x_{1,n-1} & \\
x_{2,0} & x_{2,1} & x_{2,2} & \cdots & x_{2,n-2} & & \\
\vdots & \vdots & \vdots & & & & \\
x_{n-2,0} & x_{n-2,1} & x_{n-2,2} & & & & \\
x_{n-1,0} & x_{n-1,1} & & & & & \\
x_{n,0} & & & & & &
\end{array}
$$

$$
\begin{array}{ccccccc}
x_{0,0} & x_{0,1} & x_{0,2} & x_{0,3} & x_{0,4} & x_{0,5} & x_{0,6} \\
x_{1,0} & x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} & x_{1,5} & \\
x_{2,0} & x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} & & \\
x_{3,0} & x_{3,1} & x_{3,2} & x_{3,3} & & & \\
x_{4,0} & x_{4,1} & x_{4,2} & & & & \\
x_{5,0} & x_{5,1} & & & & & \\
x_{6,0} & & & & & &
\end{array}
$$

$x_{0,0}$	$x_{0,1}$	$x_{0,2}$	$x_{0,3}$	$x_{0,4}$	$x_{0,5}$	$x_{0,6}$
$x_{1,0}$	$x_{1,1}$	$x_{1,2}$	$x_{1,3}$	$x_{1,4}$	$x_{1,5}$	
$x_{2,0}$	$x_{2,1}$	$x_{2,2}$	$x_{2,3}$	$x_{2,4}$		
$x_{3,0}$	$x_{3,1}$	$x_{3,2}$	$x_{3,3}$			
$x_{4,0}$	$x_{4,1}$	$x_{4,2}$				
$x_{5,0}$	$x_{5,1}$					
$x_{6,0}$						


x0,0 x0,1 x0,2 x0,3 x0,4 x0,5 x0,6
x1,0 x1,1 x1,2 x1,3 x1,4 x1,5
x2,0 x2,1 x2,2 x2,3 x2,4
x3,0 x3,1 x3,2 x3,3
x4,0 x4,1 x4,2
x5,0 x5,1
x6,0



$x_{0,0}$ $x_{0,1}$ $x_{0,2}$ $x_{0,3}$ $x_{0,4}$ $x_{0,5}$ $x_{0,6}$
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$x_{2,0}$ $x_{2,1}$ $x_{2,2}$ $x_{2,3}$ $x_{2,4}$
$x_{3,0}$ $x_{3,1}$ $x_{3,2}$ $x_{3,3}$
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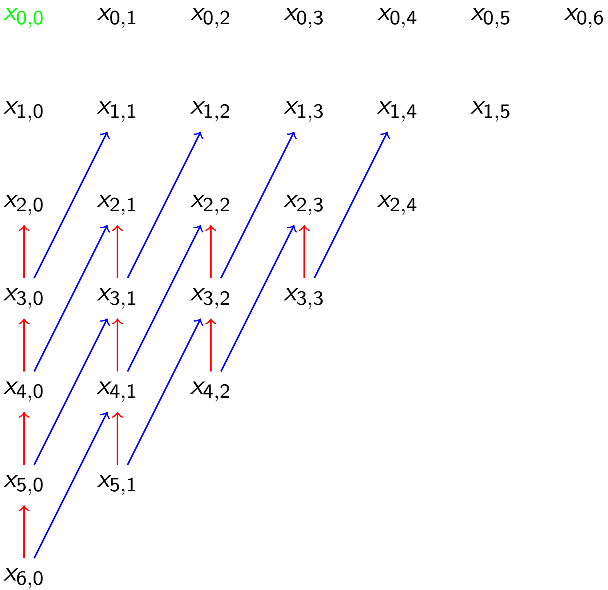
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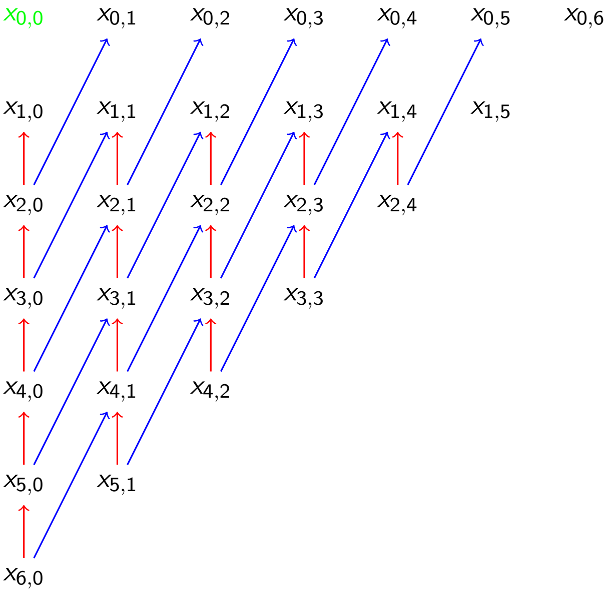
x0,0 x0,1 x0,2 x0,3 x0,4 x0,5 x0,6
x1,0 x1,1 x1,2 x1,3 x1,4 x1,5
x2,0 x2,1 x2,2 x2,3 x2,4
x3,0 x3,1 x3,2 x3,3
x4,0 x4,1 x4,2
x5,0 x5,1
x6,0

Finally Full Rank

The $2n + 1$ unknowns that remain are

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Therefore, we can solve for $x_{0,0} = \text{Holant}_\Omega$.

Outline

1 Introduction

2 Previous Work
- Dichotomy for $Z_3(\vec{G}; f)$
- Dichotomy for PI-#CSP($\mathcal{F}$)
- Dichotomy for Holant($\mathcal{F}$)

3 Current Work

4 Future Work

Definition

A signature grid $\Omega = (G, \mathcal{F})$ consists of

- a graph $G = (V, E)$,
- a set of signatures $\mathcal{F}$ with $\{0, 1\}$ inputs and a $\mathbb{C}$ output, and
- f_v is the signature on vertex v.

On input Ω, the goal is to compute

$$\text{Holant}_\Omega = \sum_{\sigma:E\to\{0,1\}} \prod_{v\in V} f_v(\sigma\mid_{E(v)}).$$

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Example

f_v = ALL-DISTINCT gives $\#\kappa$-EDGECOLORING

Current Result

Theorem (Cai, Guo, W, Xia)

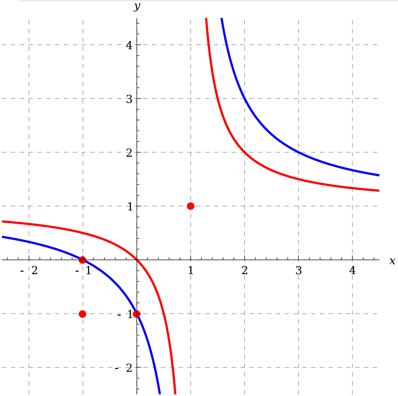
Counting κ-edge colorings over planar r-regular graphs is $\#\mathrm{P}$-hard for $\kappa \geq r \geq 3$.

[Cai, Lu, Xia 13]

- Holant$^*(f)$ with domain size $\kappa = 3$ such that
  - f has arity 3,
  - f is symmetric, and
  - f has complex weights.

Theorem (Vertigan 05)

For any $x, y \in \mathbb{C}$, the problem of computing the Tutte polynomial at (x, y) over planar *graphs is $\#\mathrm{P}$-hard unless $(x-1)(y-1) \in \{1, 2\}$ or $(x, y) \in \{(1,1), (-1,-1), (j, j^2), (j^2, j)\}$, where $j = e^{2\pi i/3}$. In each of these exceptional cases, the computation can be done in polynomial time.*



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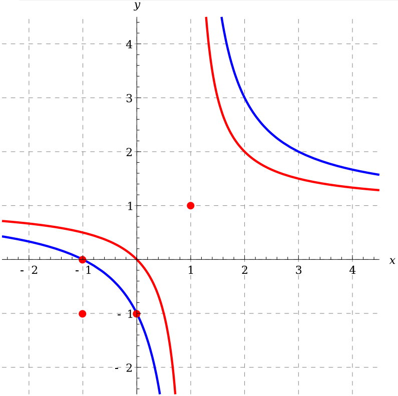

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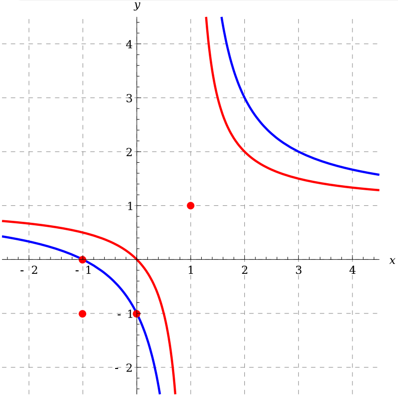

- For $\kappa = r$, reduction from Tutte$(\kappa + 1, \kappa + 1)$ for planar graphs
- For $\kappa > r$, reduction from Tutte$(1 - \kappa, 0)$ for planar graphs (i.e. counting κ-VertexColoring)

Outline

1. Introduction
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4. **Future Work**

Asymmetric Signatures

Holant over General Graphs:

[Cai, Lu, Xia 11]

- Dichotomy for Holant$^*(\mathcal{F})$ with complex weights

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- No dichotomy theorems for asymmetric signatures over planar graphs.
- True test for the universality of matchgates.

Pl-Holant($\mathcal{F}$) **with Symmetric Signatures**

My PI-#CSP($\mathcal{F}$) dichotomy with Guo is crucial.

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Need to extend #CSP^d($\mathcal{F}$) dichotomy by Huang and Lu to PI-#CSP^d($\mathcal{F}$).

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Expect the rest of the proof to be similar to previous work
(i.e. dichotomy for Holant($\mathcal{F}$) over general graphs with Cai and Guo)

Example

- Chromatic polynomial $\chi(\lambda)$
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Give back to the Tutte polynomial via consideration of regular graphs.

Previous Work:

Dichotomy theorems for

1. $Z_3(\vec{G}; f)$,
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Current Work:

$\#\mathrm{P}$-hardness of $\#\kappa$-$\textsc{EdgeColoring}$ problems.

Future Work:

- Extend all my results.
- Consider other graph polynomials.

Thank You

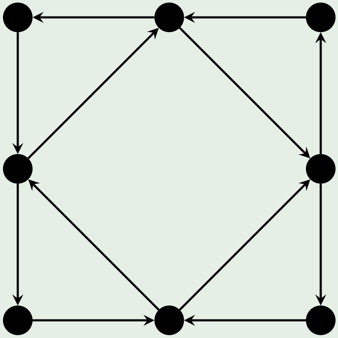
Eulerian Orientation

Definition

At each vertex in an Eulerian orientation of a graph,

in-degree equals out-degree.

Example



Theorem and Proof Overview

Theorem (Guo, W 13)

Counting Eulerian Orientations for planar 4-regular graphs is $\#\mathrm{P}$-hard.

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Strengthens a theorem from [Huang, Lu 12] to the planar setting.

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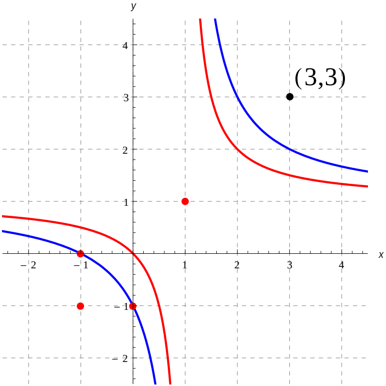
Proof.

Reduction from the evaluation of the Tutte polynomial at the point $(3, 3)$ for planar graphs:

$$\begin{aligned} \text{Pl-Tutte}(3,3) &\leq_T \quad \vdots \\ &\leq_T \#\text{Pl-4Reg-EO} \end{aligned}$$

Theorem (Vertigan 05)

For any $x, y \in \mathbb{C}$, the problem of computing the Tutte polynomial at (x, y) over planar *graphs is $\#\mathrm{P}$-hard unless $(x-1)(y-1) \in \{1, 2\}$ or $(x, y) \in \{(1,1), (-1,-1), (j, j^2), (j^2, j)\}$, where $j = e^{2\pi i/3}$. In each of these exceptional cases, the computation can be done in polynomial time.*

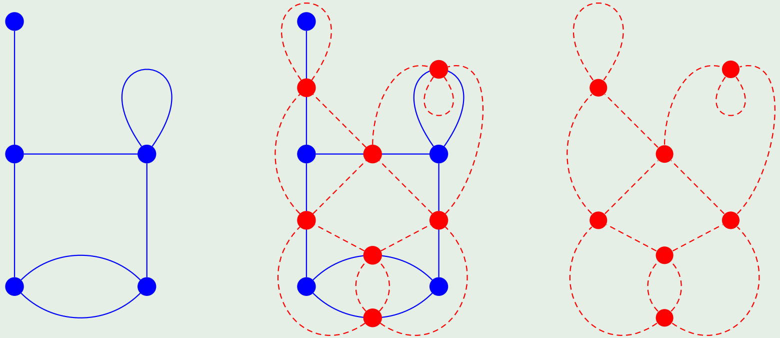


Medial Graph

Definition

For a connected plane graph G, its medial graph H has a vertex for each edge of G and two vertices in H are joined by an edge for each face of G in which their corresponding edges occur consecutively.

Example



Theorem (Las Vergnas 88)

Let G be a connected plane graph and let $\mathscr{O}(H)$ be the set of all Eulerian orientations in the medial graph H of G. Then

$$2 \cdot \text{PI-Tutte}_G(3,3) = \sum_{O \in \mathscr{O}(H)} 2^{\beta(O)},$$

where $\beta(O)$ is the number of saddle vertices in the orientation O, i.e. vertices in which the edges are oriented "in, out, in, out" in cyclic order.

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$\text{Pl-Holant}\left([0,1,0] \mid f\right)$

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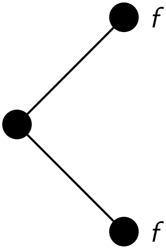

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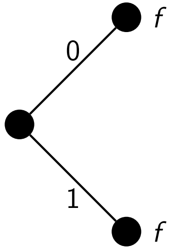

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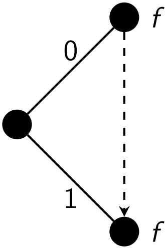

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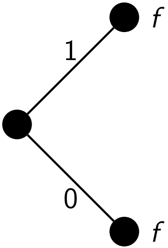

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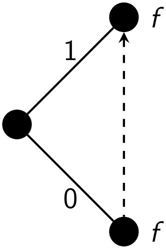

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Signature matrix:

- Let $f(w,x,y,z) = f^{wxyz}$ be an arity 4 signature
- Row index is (w,x), **BUT** the column index is (z,y) (order reversed)

$$M_f = \begin{bmatrix} f^{0000} & f^{0010} & f^{0001} & f^{0011} \\ f^{0100} & f^{0110} & f^{0101} & f^{0111} \\ f^{1000} & f^{1010} & f^{1001} & f^{1011} \\ f^{1100} & f^{1110} & f^{1101} & f^{1111} \end{bmatrix}$$

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Theorem

Counting Eulerian Orientations for planar 4-regular graphs is $\#\mathrm{P}$*-hard.*

Proof.

$$\text{Pl-Tutte}(3,3) \equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right)$$

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Theorem

Counting Eulerian Orientations for planar 4-regular graphs is $\#\mathrm{P}$*-hard.*

Proof.

$$\begin{aligned}
\text{Pl-Tutte}(3,3) &\equiv_T \text{Pl-Holant}\left([0,1,0] \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \\
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Let $Z = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$.

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$$\begin{aligned} \text{Pl-Holant}\,([0,1,0] \mid f) &\equiv_T \text{Pl-Holant}\left([0,1,0](Z^{-1})^{\otimes 2} \mid Z^{\otimes 4} f\right) \\ &\equiv_T \text{Pl-Holant}\left([1,0,1]/2 \mid 4\hat{f}\right) \\ &\equiv_T \text{Pl-Holant}(\hat{f}), \end{aligned}$$

where

$$M_{\hat{f}} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Similarly,

$$\begin{aligned}
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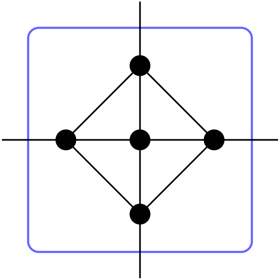
$$

Assign $[3, 0, 1, 0, 3]$ to every vertex of this gadget...



...to get a signature $32\hat{g}$ with

$$M_{\hat{g}} = \frac{1}{2}\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}.$$

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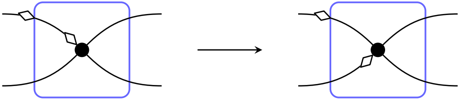
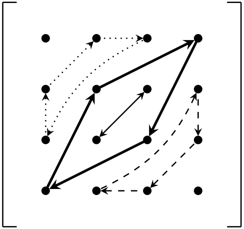
Rotationally Symmetric

$$M_{\hat{f}} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \qquad M_{\hat{g}} = \frac{1}{2} \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}$$



(a) A counterclockwise rotation.



(b) Movement of signature matrix entries under a counterclockwise rotation.

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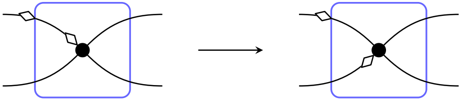

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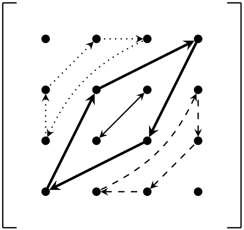

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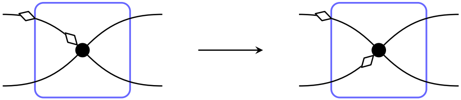

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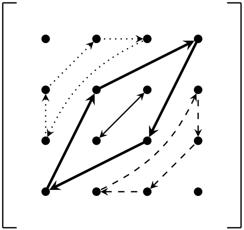

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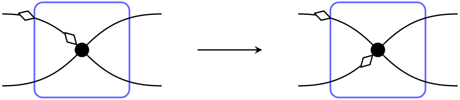

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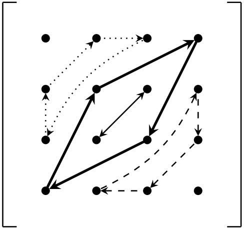
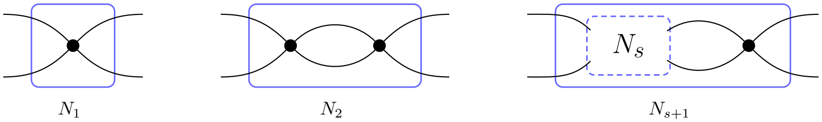


(b) Movement of signature matrix entries under a counterclockwise rotation.

Suppose that $\hat{f}$ appears n times in Ω of PI-Holant($\hat{f}$).
Construct instances Ω_s of Holant($\hat{g}$) indexed by $s \geq 1$.
Obtain Ω_s from Ω by replacing each $\hat{f}$ with N_s ($\hat{g}$ assigned to all vertices).



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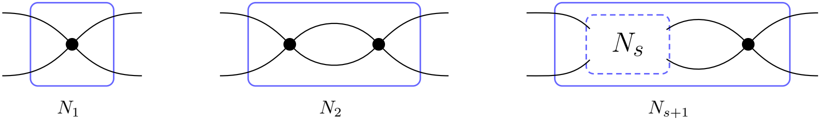

To obtain Ω_s from Ω,
we effectively replace $M_{\hat{f}}$ with $M_{N_s} = (M_{\hat{g}})^s$.

Interpolation

Let $T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$.

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and

$$M_{\hat{g}} = T\Lambda_{\hat{g}}T^{-1} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix} T^{-1}.$$

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Follows from being both rotationally symmetric and complement invariant.

Stratify

$$\Lambda_{\hat{f}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Lambda_{\hat{g}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

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We only need to consider the assignments to $\Lambda_{\hat{f}}$ that assign

- 0000 j many times,
- 0110 or 1001 k many times, and
- 1111 ℓ many times.

Let $c_{jk\ell}$ be the sum over all such assignments of the products of evaluations from T and T^{-1} but excluding $\Lambda_{\hat{f}}$ on Ω'.

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and

$$\text{PI-Holant}_{\Omega_s} = \sum_{j+k+\ell=n} (6^k 13^{\ell})^s c_{jk\ell}$$

is a full rank Vandermonde system (row index s, column index (j, k, ℓ)).

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Major proof techniques:

1. Holographic transformation
2. Gadget construction
3. Interpolation