The Complexity of Counting Edge Colorings and a Dichotomy for Some Higher Domain Holant Problems

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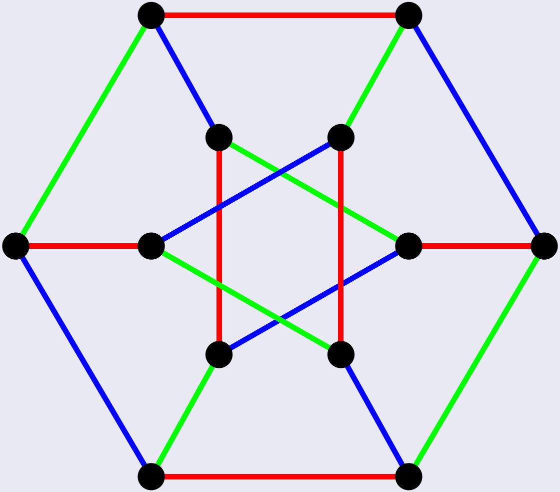
(University of Wisconsin-Madison)

Edge Coloring

Definition



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Proved in two cases:

1. $\kappa = r$, and
2. $\kappa > r$.

#κ-EdgeColoring as a Holant Problem

Let AD_3 denote the local constraint function

$$\mathrm{AD}_3(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [\kappa] \text{ are distinct} \\ 0 & \text{otherwise.} \end{cases}$$

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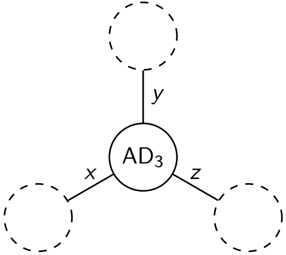
Place AD_3 at each vertex with incident edges x, y, z in a 3-regular graph G.



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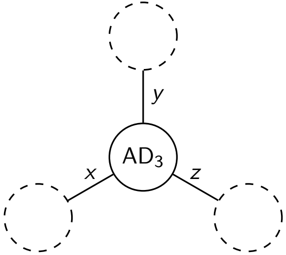

Then we define the sum of product

$$\text{Holant}_\kappa(G; \text{AD}_3) = \sum_{\sigma: E(G) \to [\kappa]} \prod_{v \in V(G)} \text{AD}_3\left(\sigma\mid_{E(v)}\right).$$

Clearly computes #κ-EdgeColoring.

Same as the partition function of the edge-coloring model.

In general, we consider all local constraint functions

$$f(x, y, z) = \begin{cases} a & \text{if } x = y = z \qquad\qquad \text{(all equal)} \\ b & \text{otherwise} \\ c & \text{if } x \neq y \neq z \neq x \qquad \text{(all distinct).} \end{cases}$$

The Holant problem is to compute

$$\text{Holant}_\kappa(G; f) = \sum_{\sigma : E(G) \to [\kappa]} \prod_{v \in V(G)} f\left(\sigma \mid_{E(v)}\right).$$

Denote f by $\langle a, b, c \rangle$.
Thus $\text{AD}_3 = \langle 0, 0, 1 \rangle$.

Dichotomy Theorem for $\text{Holant}_\kappa(-; \langle a, b, c \rangle)$

Theorem (Main Theorem)

For any $\kappa \geq 3$ and any $a, b, c \in \mathbb{C}$,
the problem of computing $\text{Holant}_\kappa(-; \langle a, b, c \rangle)$ *is in P or #P-hard,*
even when the input is restricted to planar graphs.

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Recall $\#\kappa$-EdgeColoring is the special case $\langle a, b, c\rangle = \langle 0, 0, 1\rangle$.

Let's prove the theorem for $\kappa = 3$ and $\langle a, b, c\rangle = \langle 0, 0, 1\rangle$.

For a plane graph G,

- $T(G; x, x) = m(\vec{G}_m; x)$ (Martin polynomial, [Martin '77])

Graph Polynomial Identities

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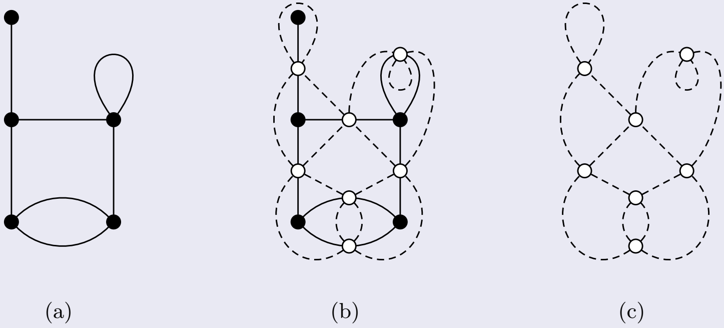
Definition (Medial Graph)



A plane graph (a), its medial graph (c), and the two graphs superimposed (b).

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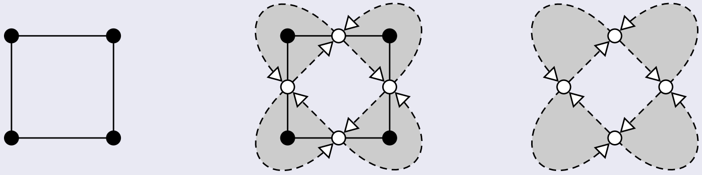
Definition (Directed Medial Graph)



(a) (b) (c)

A plane graph (a), its directed medial graph (c), and the two graphs superimposed (b).

For a plane graph G,

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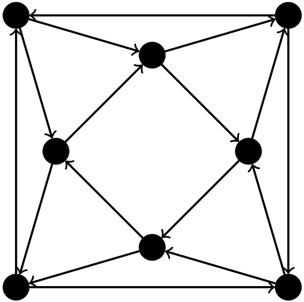
Digraph Eulerian if "in degree" = "out degree".



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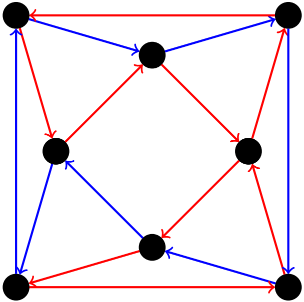

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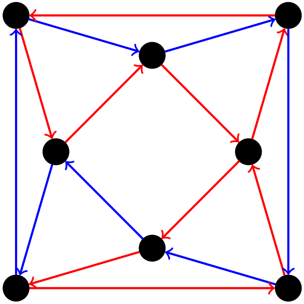

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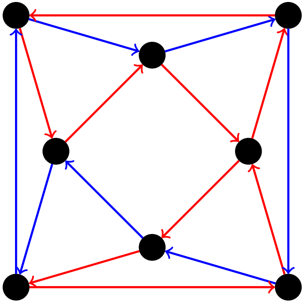
$\mu(c)$ is number of monochromatic vertices in c.



$x \geq 2$ $\mu(c) = 1$

One More Identity

Lemma

$$\sum_{c \in \pi_\kappa(\vec{G}_m)} 2^{\mu(c)} = \mathrm{Holant}_\kappa(G_m; \mathcal{E})$$

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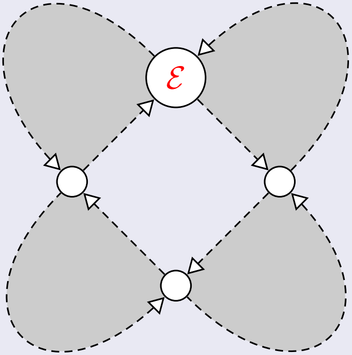
Proof.

$$\mathcal{E}\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \begin{cases} 2 & \text{if } w = x = y = z \\ 1 & \text{if } w = x \neq y = z \\ 0 & \text{if } w = y \neq x = z \\ 1 & \text{if } w = z \neq x = y \\ 0 & \text{otherwise.} \end{cases}$$



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Denote $\mathcal{E}$ by $\langle 2, 1, 0, 1, 0 \rangle$.

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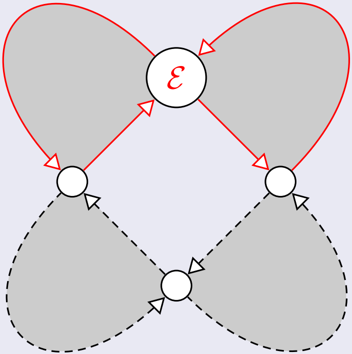

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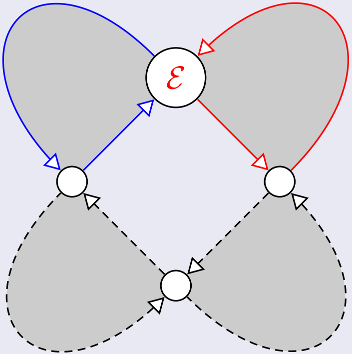

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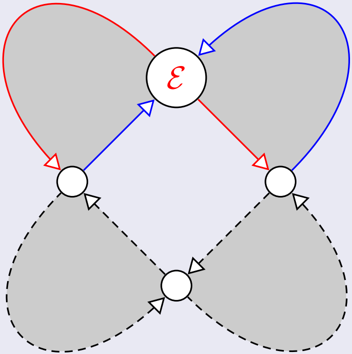

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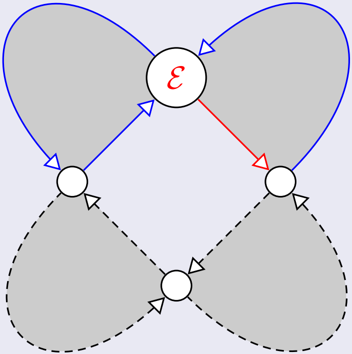

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Upshot

Corollary

For a plane graph G,

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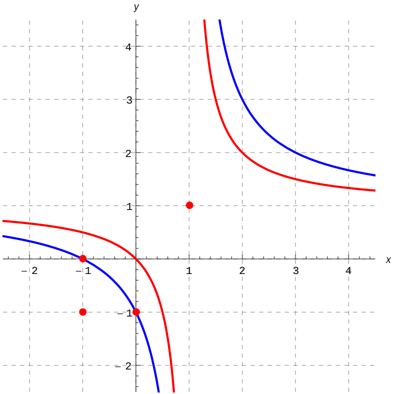
Theorem (Vertigan)

For any $x, y \in \mathbb{C}$, the problem of evaluating the Tutte polynomial at (x, y) over planar graphs is #P-hard unless $(x-1)(y-1) \in \{1, 2\}$ or $(x, y) \in \{(\pm 1, \pm 1), (\omega, \omega^2), (\omega^2, \omega)\}$, where $\omega = e^{2\pi i/3}$. In each of these exceptional cases, the computation can be done in polynomial time.



#P-hardness of #κ-EdgeColoring

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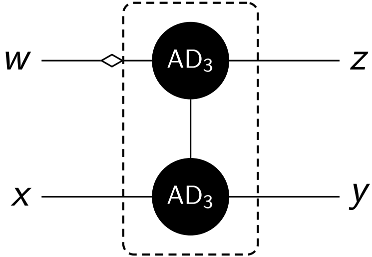
Proof for $\kappa = 3$.

Reduce from $\text{Holant}_3(-; \mathcal{E})$ to $\text{Holant}_3(-; \text{AD}_3)$ in two steps:

$$\begin{aligned} \text{Holant}_3(-; \mathcal{E}) &\leq_T \text{Holant}_3(-; \langle 0, 1, 1, 0, 0 \rangle) && \text{(polynomial interpolation)} \\ &\leq_T \text{Holant}_3(-; \text{AD}_3) && \text{(gadget construction)} \end{aligned}$$

Gadget Construction Step

$$\text{Holant}_3(G; \langle 0,1,1,0,0\rangle) = \text{Holant}_3(G'; \text{AD}_3)$$



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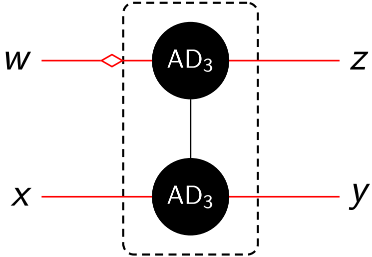

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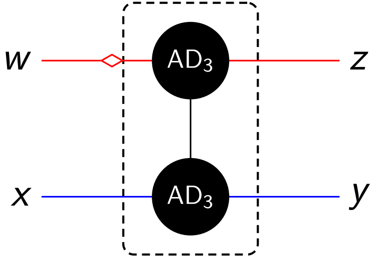

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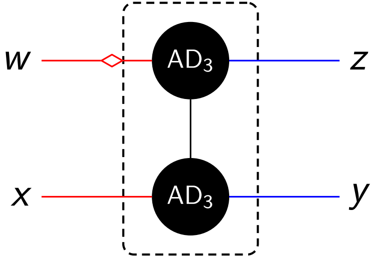

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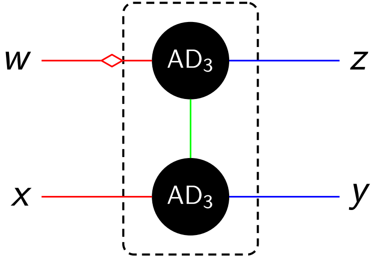

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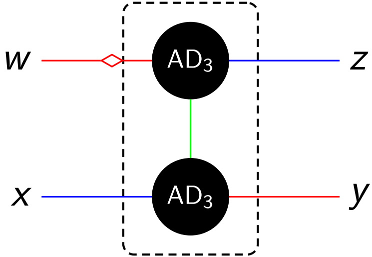

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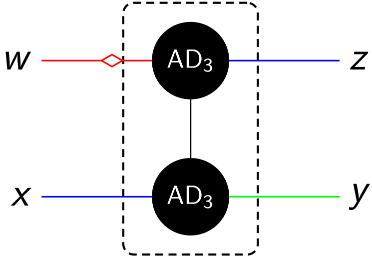

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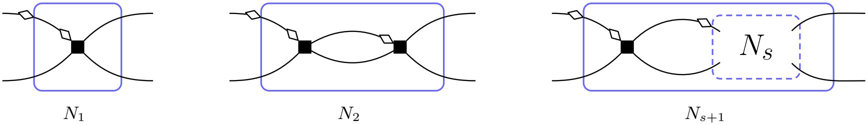
Polynomial Interpolation Step: Recursive Construction

$$\text{Holant}_3(G; \mathcal{E}) \leq_T \text{Holant}_3(G_s; \langle 0, 1, 1, 0, 0 \rangle)$$



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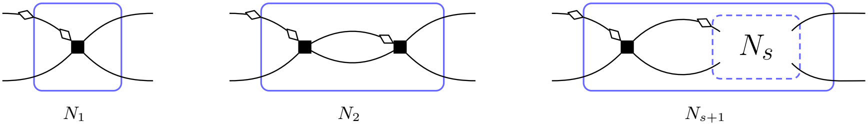

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Let f_s be the function corresponding to N_s. Then $f_s = M^s f_0$, where

$$M = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad f_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Obviously $f_1 = \langle 0, 1, 1, 0, 0 \rangle$.

Polynomial Interpolation Step: Eigenvectors and Eigenvalues

Spectral decomposition $M = P\Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Polynomial Interpolation Step: Eigenvectors and Eigenvalues

Spectral decomposition $M = P\Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $x = 2^{2s}$. Then

$$f_{2s} = P\Lambda^{2s}P^{-1}f_0 = P \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1} f_0 = \begin{bmatrix} \frac{x-1}{3} + 1 \\ \frac{x-1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

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Note $f(4) = \mathcal{E} = \langle 2, 1, 0, 1, 0 \rangle$.

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(Side note: picking $s = 1$ so that $x = 4$ only works when $\kappa = 3$.)

Polynomial Interpolation Step: The Interpolation

$$\text{Holant}_3(-; \mathcal{E}) \leq_T \text{Holant}_3(-; \langle 0, 1, 1, 0, 0 \rangle)$$

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If G has n vertices, then

$$p(G,x) = \text{Holant}_3(G;f(x)) \in \mathbb{Z}[x]$$

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Let G_s be the graph obtained by replacing every vertex in G with N_s.
Then $\text{Holant}_3(G_{2s};\langle 0,1,1,0,0\rangle) = p(G,2^{2s})$.

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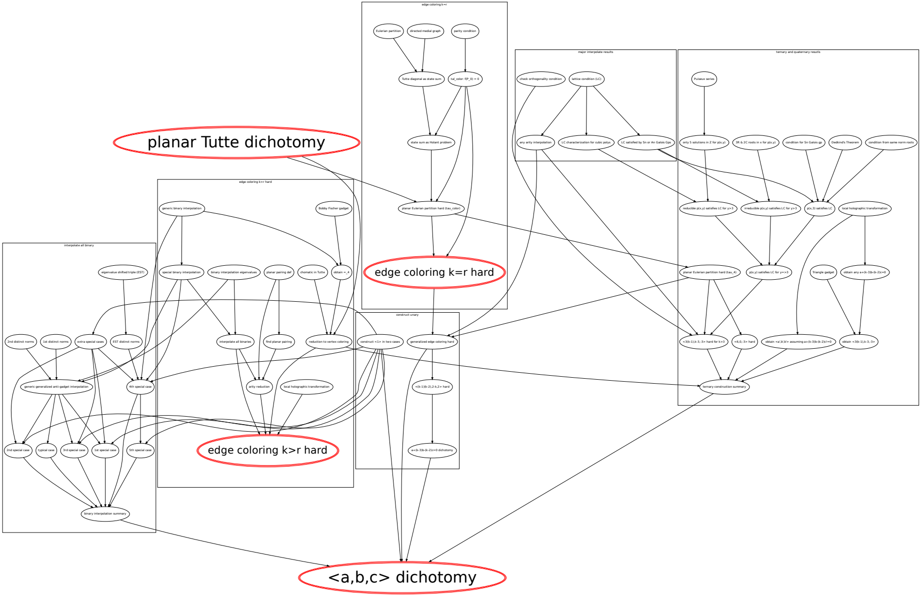
Using oracle for $\text{Holant}_3(-;\langle 0,1,1,0,0\rangle)$, evaluate $p(G,x)$ at $n+1$ distinct points $x = 2^{2s}$ for $0 \leq s \leq n$.

By polynomial interpolation, efficiently compute the coefficients of $p(G,x)$. QED.

Dichotomy of $\text{Holant}_\kappa(-; \langle a, b, c \rangle)$



Thank You

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Paper and slides available on my website:
`www.cs.wisc.edu/~tdw`