The Complexity of Counting Edge Colorings and a Dichotomy for Some Higher Domain Holant Problems

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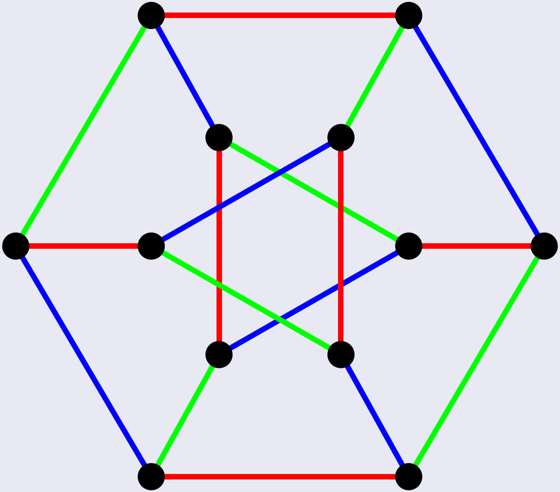
Edge Coloring

Definition



Counting Edge Colorings

Problem: $\#\kappa$-EdgeColoring
Input: A graph G.
Output: Number of edge colorings of G using at most κ colors.

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Theorem

$\#\kappa$-EdgeColoring is #P-hard over planar r-regular graphs for all $\kappa \geq r \geq 3$.

Trivially tractable when $\kappa \geq r \geq 3$ does not hold.
Parallel edges allowed (and necessary when $r > 5$).

Proved in the framework of Holant problems in two cases:

1. $\kappa = r$, and
2. $\kappa > r$.

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NON-examples: Hamiltonian cycles and spanning trees.

NOT local.

Abundance of Holant Problems

Equivalent to:

- counting read-twice constraint satisfaction problems,
- contraction of tensor networks, and
- partition function of graphical models (in Forney normal form).

Generalizes:

- simulating quantum circuits,
- counting graph homomorphisms,
- all manner of partition functions including
  - Ising model,
  - Potts model,
  - edge-coloring model.

#κ-EdgeColoring as a Holant Problem

Let AD_3 denote the local constraint function

$$\text{AD}_3(x, y, z) = \begin{cases} 1 & \text{if } x, y, z \in [\kappa] \text{ are distinct} \\ 0 & \text{otherwise.} \end{cases}$$

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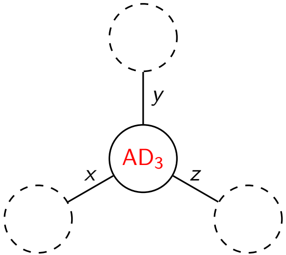
Place AD_3 at each vertex with incident edges x, y, z in a 3-regular graph G.



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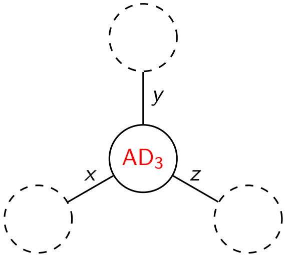

Then we evaluate the sum of product

$$\text{Holant}(G; \mathsf{AD}_3) = \sum_{\sigma : E(G) \to [\kappa]} \prod_{v \in V(G)} \mathsf{AD}_3 \left(\sigma \mid_{E(v)} \right).$$

Clearly $\text{Holant}(G; \mathsf{AD}_3)$ computes $\#\kappa$-EdgeColoring.

Some Higher Domain Holant Problems

In general, we consider all local constraint functions

$$f(x,y,z) = \begin{cases} a & \text{if } x = y = z \qquad\qquad \text{(all equal)} \\ b & \text{otherwise} \\ c & \text{if } x \neq y \neq z \neq x \qquad \text{(all distinct).} \end{cases}$$

The Holant problem is to compute

$$\text{Holant}_{\kappa}(G; f) = \sum_{\sigma : E(G) \to [\kappa]} \prod_{v \in V(G)} f\left(\sigma \mid_{E(v)}\right).$$

Denote f by $\langle a, b, c \rangle$.
Thus $\text{AD}_3 = \langle 0, 0, 1 \rangle$.

Dichotomy Theorem for $\text{Holant}_\kappa(-; \langle a, b, c \rangle)$

Theorem (Main Theorem)

For any $\kappa \geq 3$ *and any* $a, b, c \in \mathbb{C}$,
the problem of computing $\text{Holant}_\kappa(-; \langle a, b, c \rangle)$ *is in P or #P-hard,*
even when the input is restricted to planar graphs.

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Recall $\#\kappa$-EdgeColoring is the special case $\langle a, b, c \rangle = \langle 0, 0, 1 \rangle$.

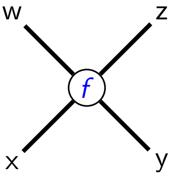
Let's prove the theorem for $\kappa = 3$ and $\langle a, b, c \rangle = \langle 0, 0, 1 \rangle$.

Hardness of $\text{Holant}_3(-; \text{AD}_3)$ proved by the following reduction chain:

$$\begin{aligned} \#\text{P} &\leq_T \text{Holant}_3(-; \langle 2, 1, 0, 1, 0 \rangle) \\ &\leq_T \text{Holant}_3(-; \langle 0, 1, 1, 0, 0 \rangle) \\ &\leq_T \text{Holant}_3(-; \text{AD}_3) \end{aligned}$$

$\langle a, b, c, d, e \rangle$ denotes an arity-4 function f

$$f\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \begin{cases} a & \text{if } w = x = y = z \\ b & \text{if } w = x \neq y = z \\ c & \text{if } w = y \neq x = z \\ d & \text{if } w = z \neq x = y \\ e & \text{otherwise.} \end{cases}$$



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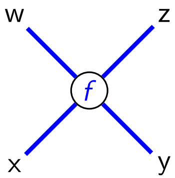

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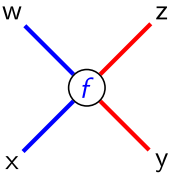

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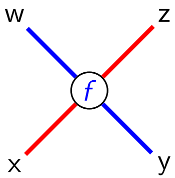

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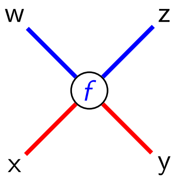

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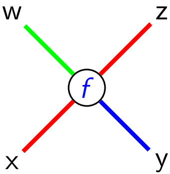

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- Third reduction: Via a gadget construction.

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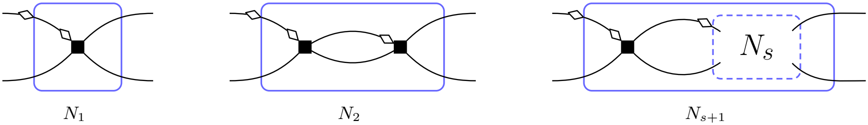
Polynomial Interpolation Step: Recursive Construction

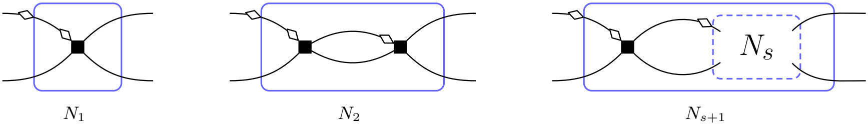
$$\text{Holant}_3(G; \langle 2, 1, 0, 1, 0 \rangle) \leq_T \text{Holant}_3(G_s; \langle 0, 1, 1, 0, 0 \rangle)$$



Vertices are assigned $\langle 0, 1, 1, 0, 0 \rangle$.

Polynomial Interpolation Step: Recursive Construction

$$\text{Holant}_3(G; {\color{blue}\langle 2,1,0,1,0\rangle}) \le_T \text{Holant}_3(G_s; {\color{blue}\langle 0,1,1,0,0\rangle})$$



Vertices are assigned ${\color{blue}\langle 0,1,1,0,0\rangle}$.

Let f_s be the function corresponding to N_s. Then $f_s = M^s f_0$, where

$$M = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad f_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Obviously $f_1 = {\color{blue}\langle 0,1,1,0,0\rangle}$.

Polynomial Interpolation Step: Eigenvectors and Eigenvalues

Spectral decomposition $M = P\Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & -2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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Let $x = 2^{2s}$. Then

$$f_{2s} = P\Lambda^{2s}P^{-1}f_0 = P \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1}f_0 = \begin{bmatrix} \frac{x-1}{3} + 1 \\ \frac{x-1}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

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Note $f(4) = \langle 2, 1, 0, 1, 0 \rangle$.

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Note $f(4) = \langle 2, 1, 0, 1, 0 \rangle$.
(Side note: picking $s = 1$ so that $x = 4$ only works when $\kappa = 3$.)

Polynomial Interpolation Step: The Interpolation

$$\text{Holant}_3(-; \langle 2, 1, 0, 1, 0 \rangle) \leq_T \text{Holant}_3(-; \langle 0, 1, 1, 0, 0 \rangle)$$

Polynomial Interpolation Step: The Interpolation

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If G has n vertices, then

$$p(G, x) = \text{Holant}_3(G; f(x)) \in \mathbb{Z}[x]$$

has degree n.

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Let G_{2s} be the graph obtained by replacing every vertex in G with N_{2s}. Then $\text{Holant}_3(G_{2s}; \langle 0,1,1,0,0\rangle) = p(G, 2^{2s})$.

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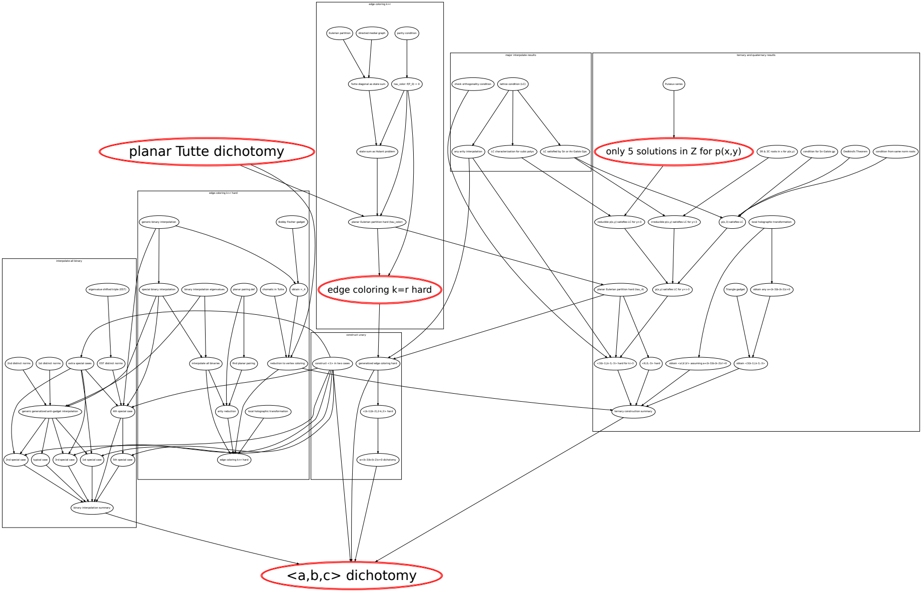
Using oracle for $\text{Holant}_3(-; \langle 0,1,1,0,0 \rangle)$, evaluate $p(G, x)$ at $n + 1$ distinct points $x = 2^{2s}$ for $0 \leq s \leq n$.

By polynomial interpolation, efficiently compute the coefficients of $p(G, x)$. QED.

Dichotomy of $\text{Holant}_\kappa(-; \langle a, b, c\rangle)$



Thank You

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Paper and slides available on my website:
`www.cs.wisc.edu/~tdw`