Advances in the Computational Complexity of Holant Problems

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Final Oral Defense
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Outline

1. **Introduction**
2. **Dichotomy Theorems**
   - Dichotomy for $Z(f)$ over Planar 3-Regular Directed Graphs
   - Dichotomy for $\#\mathrm{CSP}(\mathcal{F})$ over Planar Graphs
   - Dichotomy for Holant$(\mathcal{F})$ over General Graphs
   - Dichotomy for Holant$_\kappa(f)$ over Planar 3-Regular Graphs
3. **Example Proofs of Hardness**
   - Common Reduction
   - #EulerianOrientation over Planar 4-Regular Graphs
   - #3-EdgeColoring over Planar 3-Regular Graphs
4. **Summary**

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4. **Summary**

Complexity of Counting Problems

Counting problem

- **Input:** Graph
- **Output:** Number

Framework of problems

Dichotomy Theorem

- Every problem in the framework is either easy or hard
  (i.e. computable in polynomial time or #**P-hard**).

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#VertexCover

Definition

A vertex cover of a graph is a subset of vertices such that each edge is incident to at least one vertex in the subset.

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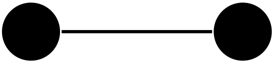

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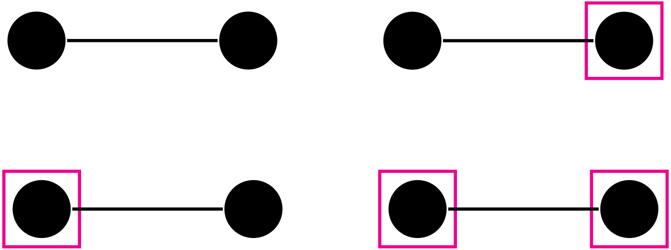

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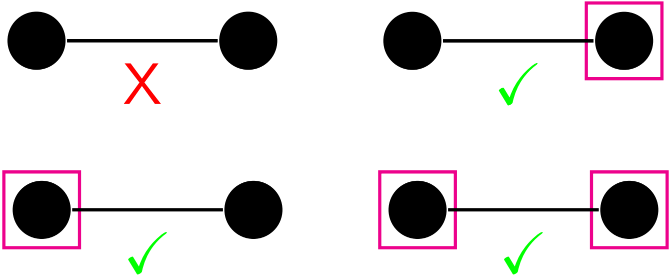
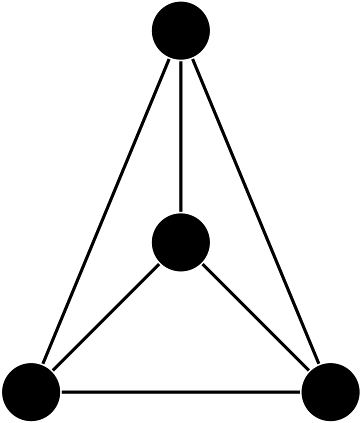


- $G = (V, E)$



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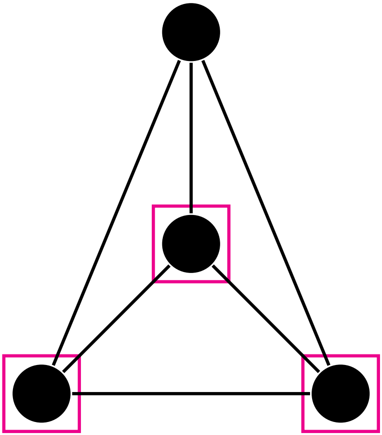
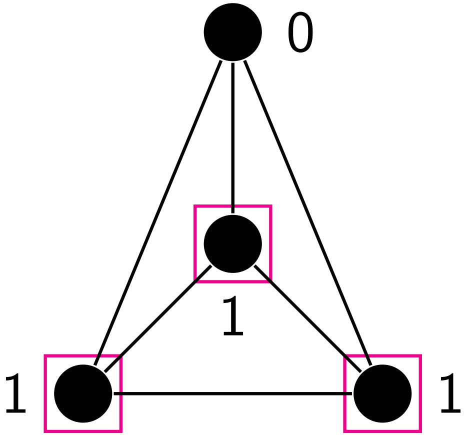


Systematic Approach to #VertexCover

- $G = (V, E)$
- $\sigma : V \to \{0, 1\}$



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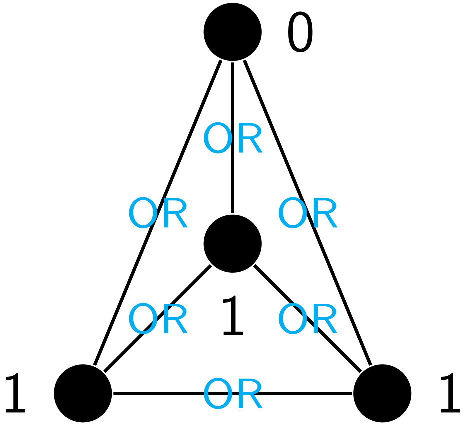

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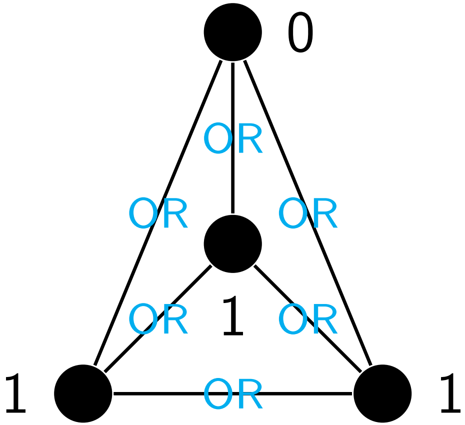

$$\prod_{(u,v)\in E} \mathrm{OR}(\sigma(u), \sigma(v)) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

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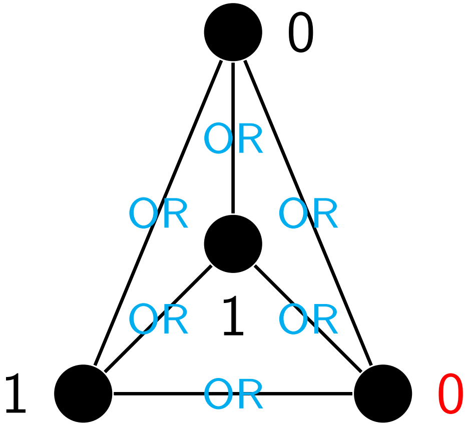

$$\prod_{(u,v)\in E} \mathrm{OR}(\sigma(u), \sigma(v)) = 1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 0$$

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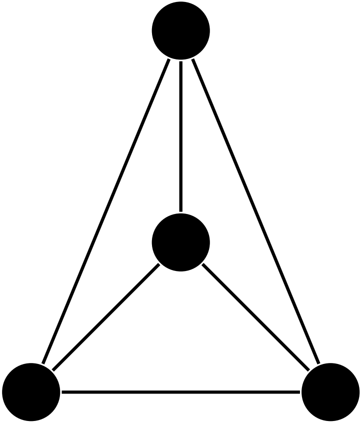
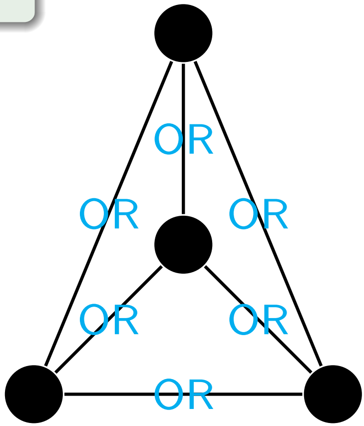


$$\#\text{VertexCover}(G) = \sum_{\sigma: V \to \{0,1\}} \prod_{(u,v) \in E} \text{OR}(\sigma(u), \sigma(v))$$

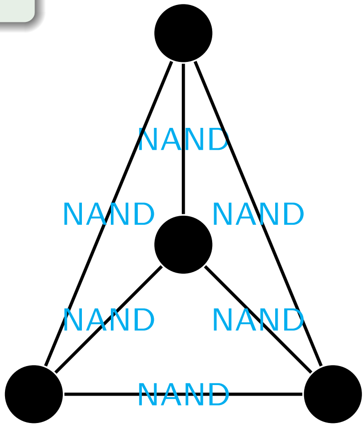
Example

- OR corresponds to #VertexCover



Example

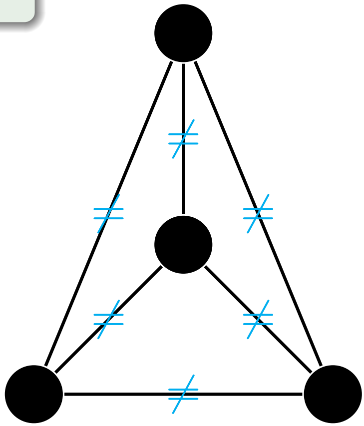
- OR corresponds to #VertexCover
- NAND corresponds to #IndependentSet



Other Edge Constraints

Example

- OR corresponds to #VertexCover
- NAND corresponds to #IndependentSet
- $\neq$ corresponds to #Bipartition



Other Edge Constraints

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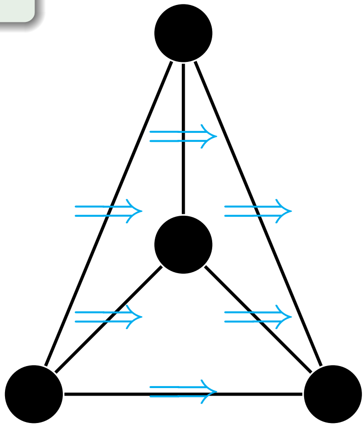

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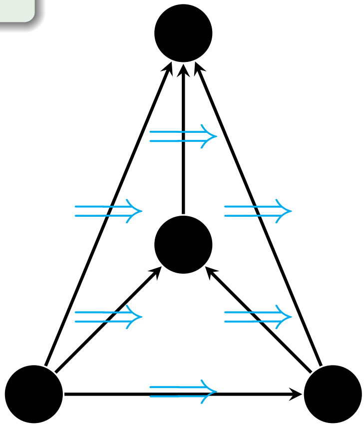
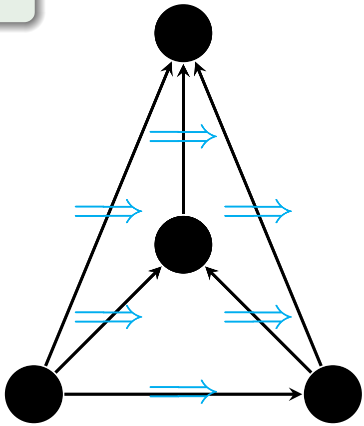


Example

- OR corresponds to #VertexCover
- NAND corresponds to #IndependentSet
- $\neq$ corresponds to #Bipartition
- $\implies$ corresponds to #UpperSet



Generalize

$$\sum_{\sigma:V\to\{0,1\}}\prod_{(u,v)\in E}\mathsf{OR}(\sigma(u),\sigma(v))$$

Generalize

$$\sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E} \mathrm{OR}(\sigma(u), \sigma(v))$$

Input		Output
p	q	$\mathrm{OR}(p,q)$
0	0	0
0	1	1
1	0	1
1	1	1

Generalize

$$\sum_{\sigma:V\to\{0,1\}} \prod_{(u,v)\in E} \textcolor{red}{f}(\sigma(u),\sigma(v))$$

Input		Output
p	q	$\textcolor{cyan}{\mathrm{OR}}(p,q)$
0	0	0
0	1	1
1	0	1
1	1	1

Input		Output
p	q	$\textcolor{red}{f}(p,q)$
0	0	w
0	1	x
1	0	y
1	1	z

where $w, x, y, z \in \mathbb{C}$

Generalize

Partition Function:

$$Z(\vec{G}; f) = \sum_{\sigma: V \to \{0,1\}} \prod_{(u,v) \in E} f(\sigma(u), \sigma(v))$$

Input		Output
p	q	$\mathrm{OR}(p,q)$
0	0	0
0	1	1
1	0	1
1	1	1

Input		Output
p	q	$f(p,q)$
0	0	w
0	1	x
1	0	y
1	1	z

where $w, x, y, z \in \mathbb{C}$

Theorem (Cai, Kowalczyk, W 12)

Over planar 3-regular $\vec{G}$,

$$Z(\vec{G}; f) = \sum_{\sigma: V \to \{0,1\}} \prod_{(u,v) \in E} f(\sigma(u), \sigma(v))$$

is either computable in polynomial time or $\#$**P-hard**.

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Explicit form for tractable cases.

Relation to Previous Work

Previous work:

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    - $\{0,1\}$ [Cai, Lu, Xia 08]
    - $\{0,1,-1\}$ [Kowalczyk 09]
    - $\mathbb{R}$ [Cai, Lu, Xia 09]
    - $\mathbb{C}$ [Kowalczyk, Cai 10]
  - k-regular graphs with weights in
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Our work:

- $f(0,1) \neq f(1,0)$ (i.e. directed graphs)
  - 3-regular graphs with weights in
    - $\mathbb{C}$

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Counting Constraint Satisfaction Problems (#CSP)

A set $\mathcal{F}$ of functions defines the counting problem $\#\text{CSP}(\mathcal{F})$.

Example

SAT	has	$\mathcal{F} = \{\text{OR}_n \mid n \geq 1\} \cup \{\text{NOT-EQUAL}_2\}$
3SAT	has	$\mathcal{F} = \{\text{OR}_3, \text{NOT-EQUAL}_2\}$
1-in-3SAT	has	$\mathcal{F} = \{\text{EXACT-ONE}_3, \text{NOT-EQUAL}_2\}$
NAE-3SAT	has	$\mathcal{F} = \{\text{NOT-ALL-EQUAL}_3, \text{NOT-EQUAL}_2\}$
Monotone SAT	has	$\mathcal{F} = \{\text{OR}_n \mid n \geq 1\}$
Monotone 3SAT	has	$\mathcal{F} = \{\text{OR}_3\}$
Monotone 1-in-3SAT	has	$\mathcal{F} = \{\text{EXACT-ONE}_3\}$
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Example

Problem: $\#\text{CSP}(\mathcal{F})$ with $\mathcal{F} = \{\text{EVEN-PARITY}_3, \text{MAJORITY}_3, \text{OR}_3\}$
Input: $\text{EVEN-PARITY}_3(x, y, z) \wedge \text{MAJORITY}_3(x, y, z) \wedge \text{OR}_3(x, y, z)$
Output: 3

Planar Hyper-graph

$$\text{EVEN-PARITY}_3(x, y, z) \wedge \text{MAJORITY}_3(x, y, z) \wedge \text{OR}_3(x, y, z)$$

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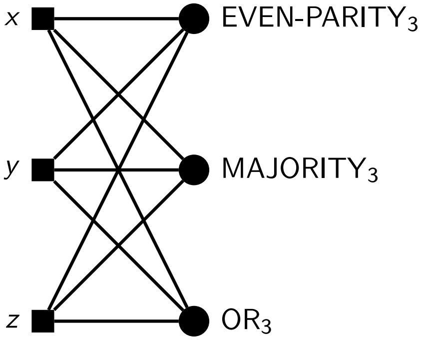

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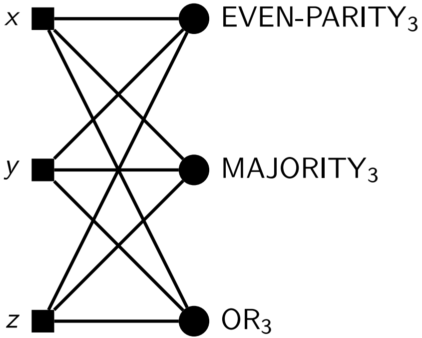
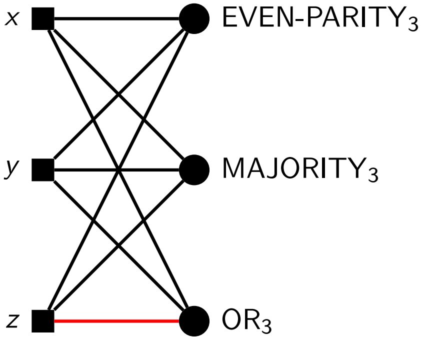


NOT Planar

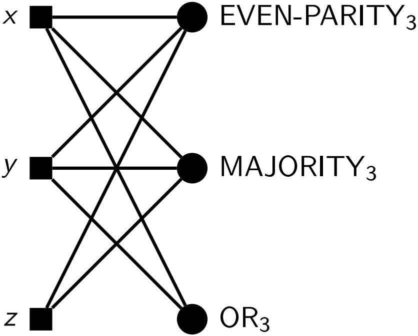
$$\text{EVEN-PARITY}_3(x, y, z) \wedge \text{MAJORITY}_3(x, y, z) \wedge \text{OR}_3(x, y, z)$$



NOT Planar

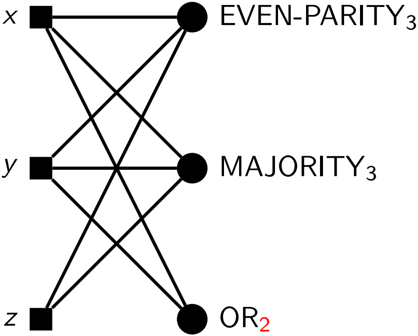
Planar Hyper-graph

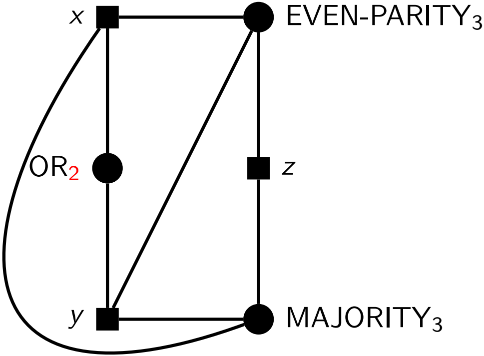
$$\text{EVEN-PARITY}_3(x, y, z) \wedge \text{MAJORITY}_3(x, y, z) \wedge \text{OR}_{\color{red}2}(x, y)$$



Planar Hyper-graph

$$\text{EVEN-PARITY}_3(x, y, z) \wedge \text{MAJORITY}_3(x, y, z) \wedge \text{OR}_2(x, y)$$





Planar

Problem: $\#\mathrm{CSP}(\mathcal{F})$

Input: Hyper-graph $G = (V, E)$ with $f_v \in \mathcal{F}$ for all $v \in V$.

- Set V of vertices (i.e. constraints)
- Set E of hyper-edges (i.e. variables)

Output:

$$\sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v \left(\sigma \mid_{E(v)}\right),$$

where $E(v)$ is the set of hyper-edges containing v.

Generalize

Problem: #CSP($\mathcal{F}$)

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Definition

A symmetric function is invariant under any permutation of its input.

Example

OR AND EVEN-PARITY MAJORITY EQUALITY

Theorem (Guo, W 13)

Let $\mathcal{F}$ be a set of symmetric functions with Boolean inputs and complex outputs.

Then over planar graphs, $\#\mathrm{CSP}(\mathcal{F})$ is #**P-hard** *unless*

1. $\mathcal{F} \subseteq \mathscr{P}$ *(Propagation),*
2. $\mathcal{F} \subseteq \mathscr{A}$ *(reduction to computing Gauss sums), or*
3. $\mathcal{F} \subseteq \mathscr{M}$ *(reduction to computing weighted perfect matchings),*

which are computable in polynomial time.

Relation to Previous Work

Special cases

- real weights [Cai, Lu, Xia 10]
- single binary function [Cai, Kowalczyk 10]

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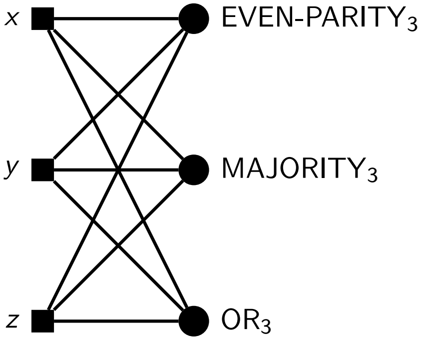
Holant Problems

Let $\mathcal{F}$ be a set of functions.

Then $\mathcal{F}$ defines the counting problem Holant($\mathcal{F}$),
which is equivalent to READ-TWICE $\#$CSP($\mathcal{F}$).

Read-Twice #CSP

$$\text{EVEN-PARITY}_3(x, y, z) \wedge \text{MAJORITY}_3(x, y, z) \wedge \text{OR}_3(x, y, z)$$



Read-Twice #CSP

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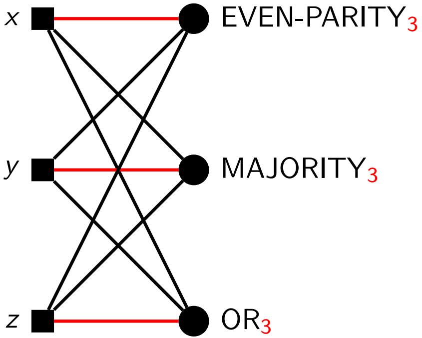
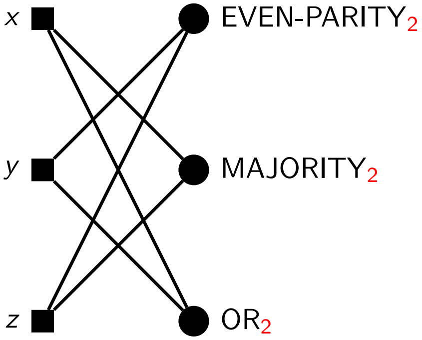
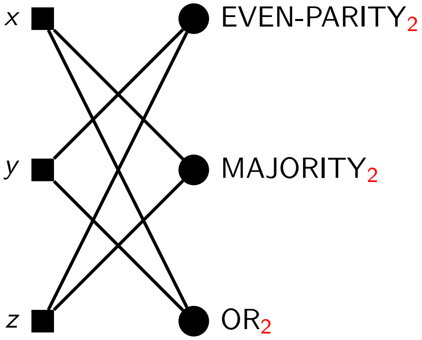
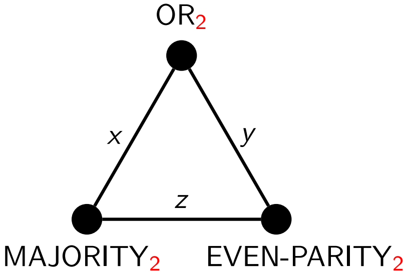


$$\text{EVEN-PARITY}_2(\quad y, z) \wedge \text{MAJORITY}_2(x, \quad z) \wedge \text{OR}_2(x, y \quad)$$



$$\text{EVEN-PARITY}_{2}(\quad y, z) \wedge \text{MAJORITY}_{2}(x, \quad z) \wedge \text{OR}_{2}(x, y \quad)$$





Problem: Holant($\mathcal{F}$)

Input: ~~Hyper~~ Graph $G = (V, E)$ with $f_v \in \mathcal{F}$ for all $v \in V$.

- Set V of vertices
- Set E of ~~hyper~~ edges

Output:

$$\text{Holant}(G; \mathcal{F}) = \sum_{\sigma : E \to \{0,1\}} \prod_{v \in V} f_v \left(\sigma \mid_{E(v)}\right),$$

where $E(v)$ is the set of ~~hyper~~ edges containing v.

Generalize

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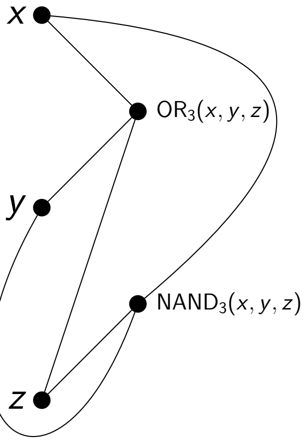
Example

Holant(G; $\mathcal{F}$) counts

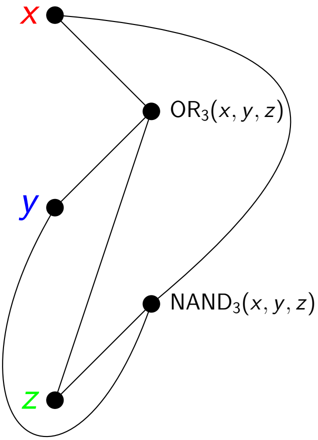
$$\begin{cases} \text{matchings} & \text{in } G \text{ when } f_v = \text{AT-MOST-ONE;} \\ \text{perfect matchings} & \text{in } G \text{ when } f_v = \text{EXACT-ONE;} \\ \text{cycle covers} & \text{in } G \text{ when } f_v = \text{EXACT-TWO;} \\ \text{edge covers} & \text{in } G \text{ when } f_v = \text{OR.} \end{cases}$$

Holographic Transformation



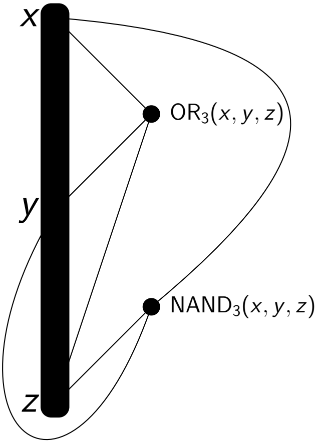
Holographic Transformation

$(1\ 0\ 0\ 1)_{x}\quad(1\ 0\ 0\ 1)_{y}\quad(1\ 0\ 0\ 1)_{z}$



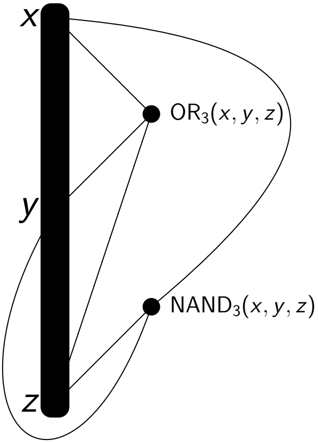
Holographic Transformation

$(1\ 0\ 0\ 1)_x \quad (1\ 0\ 0\ 1)_y \quad (1\ 0\ 0\ 1)_z$



Holographic Transformation

$(1\ 0\ 0\ 1)_x \otimes (1\ 0\ 0\ 1)_y \otimes (1\ 0\ 0\ 1)_z$



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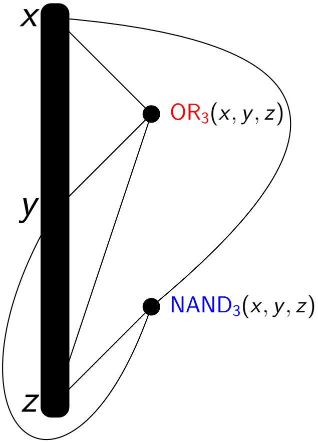
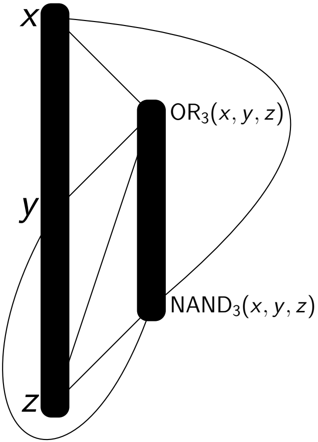


$$\begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}_{\mathrm{OR}_3}$$

$$\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\0\end{pmatrix}_{\mathrm{NAND}_3}$$

$(1\ 0\ 0\ 1)_x \otimes (1\ 0\ 0\ 1)_y \otimes (1\ 0\ 0\ 1)_z$



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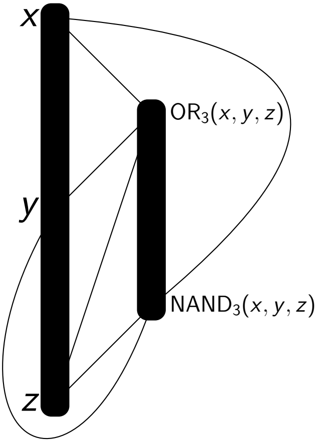
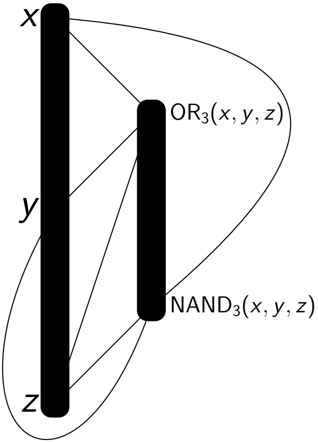


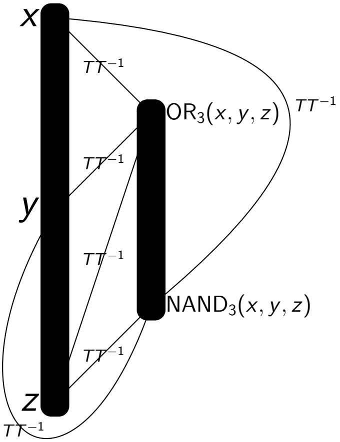
$$\underbrace{\begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}}_{\mathrm{OR}_3} \otimes \underbrace{\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\0\end{pmatrix}}_{\mathrm{NAND}_3}$$

$$(1\ 0\ 0\ 1)_x \otimes (1\ 0\ 0\ 1)_y \otimes (1\ 0\ 0\ 1)_z \cdot \begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}_{\mathrm{OR}_3} \otimes \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\0\end{pmatrix}_{\mathrm{NAND}_3}$$



Holographic Transformation

$$(1\ 0\ 0\ 1)_x \otimes (1\ 0\ 0\ 1)_y \otimes (1\ 0\ 0\ 1)_z\, T^{\otimes 6} (T^{-1})^{\otimes 6}$$



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Holographic Transformation

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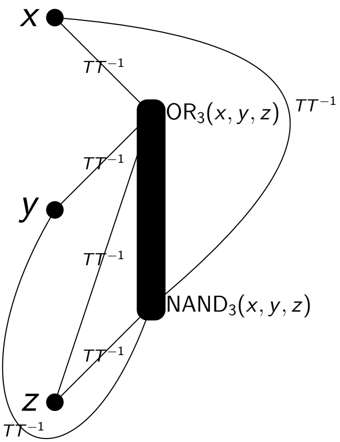
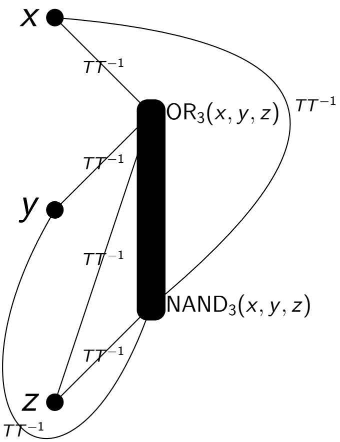


$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{\mathrm{OR}_3} \otimes \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}_{\mathrm{NAND}_3}$$

Holographic Transformation

$$(1\ 0\ 0\ 1)_x \otimes (1\ 0\ 0\ 1)_y \otimes (1\ 0\ 0\ 1)_z \left(T^{\otimes 2}\right)^{\otimes 3} (T^{-1})^{\otimes 6} \begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}_{\mathrm{OR}_3} \otimes \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\0\end{pmatrix}_{\mathrm{NAND}_3}$$



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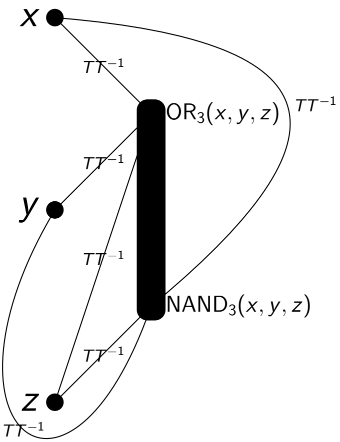
$$(1\ 0\ 0\ 1)_x T^{\otimes 2} \otimes (1\ 0\ 0\ 1)_y T^{\otimes 2} \otimes (1\ 0\ 0\ 1)_z T^{\otimes 2} (T^{-1})^{\otimes 6} \quad \begin{pmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}_{\mathrm{OR}_3} \otimes \begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\0\end{pmatrix}_{\mathrm{NAND}_3}$$



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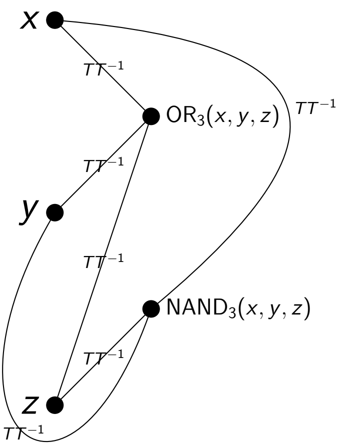

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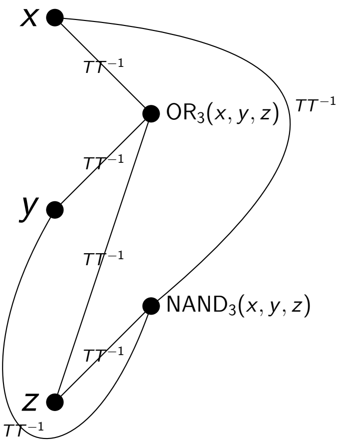
Holographic Transformation

$$(1\ 0\ 0\ 1)_x\, T^{\otimes 2} \otimes (1\ 0\ 0\ 1)_y\, T^{\otimes 2} \otimes (1\ 0\ 0\ 1)_z\, T^{\otimes 2} \left((T^{-1})^{\otimes \color{red}{3}} \right)^{\otimes \color{red}{2}}$$



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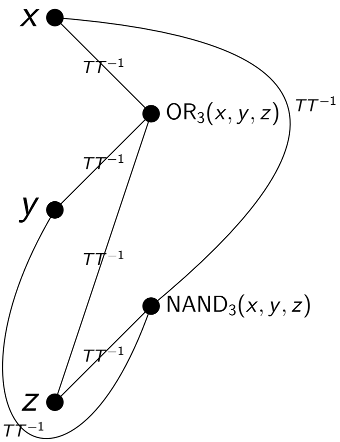

$$(T^{-1})^{\otimes 3} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{\mathrm{OR}_3} \otimes (T^{-1})^{\otimes 3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}_{\mathrm{NAND}_3}$$

Tractable Cases

- Arity 1

Tractable Cases

- Arity 1
- Arity 2

Tractable Cases

- Arity 1
- Arity 2
- #CSP tractable cases
    - $\mathscr{P}$
    - $\mathscr{A}$

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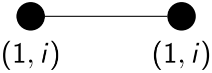

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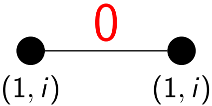

$1 \cdot 1$

Tractable Cases

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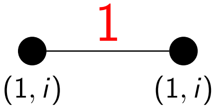

$1 \cdot 1 + i \cdot i$

Tractable Cases

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- #CSP tractable cases
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$$1 \cdot 1 + i \cdot i = 0$$

Theorem (Cai, Guo, W 13)

Let f be a symmetric function with Boolean inputs and complex outputs.

Then $\mathrm{Holant}(f)$ *is* $\#$**P-hard** *unless*

1. f *is unary,*
2. f *is binary,*
3. f *is* $\mathscr{P}$*-transformable,*
4. f *is* $\mathscr{A}$*-transformable, or*
5. f *is vanishing,*

which are computable in polynomial time.

Dichotomy for Set of Functions

Theorem (Cai, Guo, W 13)

Let $\mathcal{F}$ be a set of symmetric functions with Boolean inputs and complex outputs.

Then Holant($\mathcal{F}$) *is* #**P-hard** *unless*

1. $\mathcal{F} \subseteq \{\textit{unary}\} \cup \{\textit{binary}\}$,
2. $\mathcal{F}$ *is* $\mathscr{P}$*-transformable*,
3. $\mathcal{F}$ *is* $\mathscr{A}$*-transformable*,
4. $\mathcal{F} \subseteq \{\textit{vanishing}\} \cup \{\textit{special binary}\}$, *or*
5. $\mathcal{F} \subseteq \{\textit{"highly" vanishing}\} \cup \{\textit{special binary}\} \cup \{\textit{unary}\}$,

which are computable in polynomial time.

Relation to Previous Work

Single signature:

- Holant(ternary) with complex weights [Cai, Huang, Lu 10]
- Holant(binary $\mid =_k$) with complex weights [Cai, Kowalczyk 11]

Signature set:

- Holant$^c(\mathcal{F})$ with complex weights [Cai, Huang, Lu 10]
- $\#$CSP$^d(\mathcal{F})$ with complex weights [Huang, Lu 12]
- Holant$(\mathcal{F})$ with real weights [Huang, Lu 12]

Outline

1 Introduction

2 **Dichotomy Theorems**
- Dichotomy for $Z(f)$ over Planar 3-Regular Directed Graphs
- Dichotomy for $\#$CSP($\mathcal{F}$) over Planar Graphs
- Dichotomy for Holant($\mathcal{F}$) over General Graphs
- Dichotomy for $\text{Holant}_{\kappa}(f)$ over Planar 3-Regular Graphs

3 Example Proofs of Hardness
- Common Reduction
- $\#$EulerianOrientation over Planar 4-Regular Graphs
- $\#$3-EdgeColoring over Planar 3-Regular Graphs

4 Summary

Problem: Holant$_\kappa(\mathcal{F})$

Input: Graph $G = (V, E)$ with $f_v \in \mathcal{F}$ for all $v \in V$.

- Set V of vertices
- Set E of edges

Output:

$$\text{Holant}_\kappa(G; \mathcal{F}) = \sum_{\substack{\cancel{\sigma: E \to \{0,1\}} \\ \sigma: E \to [\kappa]}} \prod_{v \in V} f_v \left(\sigma \mid_{E(v)}\right),$$

where $E(v)$ is the set of edges containing v.

Generalize

Problem: Holant$_\kappa(\mathcal{F})$
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where $E(v)$ is the set of edges containing v.

Example

Holant$_\kappa(G; \mathcal{F})$ counts edge colorings when $f_v =$ ALL-DISTINCT.

Theorem (Cai, Guo, W 14)

Over planar 3-regular graphs, $\mathrm{Holant}_{\kappa}(f)$ *is either computable in polynomial time or* #**P-hard**, *where*

$$f(x,y,z) = \begin{cases} a & \text{if } x = y = z \qquad \text{(all equal)} \\ b & \text{otherwise} \\ c & \text{if } x \neq y \neq z \neq x \qquad \text{(all distinct)} \end{cases}$$

with $a, b, c \in \mathbb{C}$.

Explicit form for tractable cases.

Relation to Previous Work

[Cai, Lu, Xia 13]

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[Cai, Lu, Xia 13]

- arity$(f) = 3$
- f is symmetric
- $\mathbb{C}$ weights

Our work

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Relation to Previous Work

[Cai, Lu, Xia 13]

- $\text{arity}(f) = 3$
- f is symmetric
- $\mathbb{C}$ weights
- all unaries available

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- $\text{arity}(f) = 3$
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- no unaries available
- $\kappa \geq 3$

Relation to Previous Work

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Our work

- arity$(f) = 3$
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- $\kappa \geq 3$
- assume domain invariance

Outline

Reduce From Tutte Polynomial

Definition

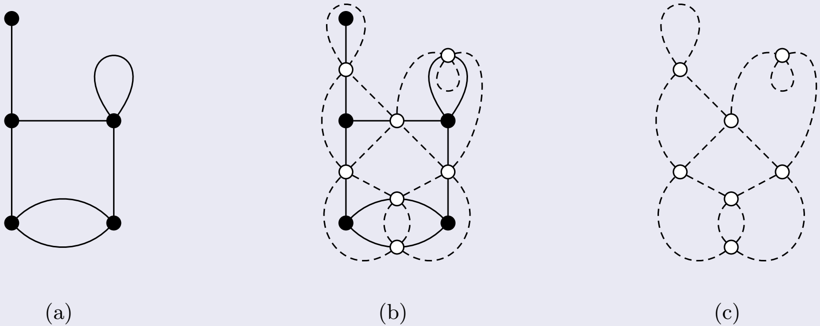
The Tutte polynomial of an undirected graph G is

$$T(G;x,y) = \begin{cases} 1 & E(G) = \emptyset, \\ xT(G \setminus e; x, y) & e \in E(G) \text{ is a bridge,} \\ yT(G \setminus e; x, y) & e \in E(G) \text{ is a loop,} \\ T(G \setminus e; x, y) + T(G/e; x, y) & \text{otherwise,} \end{cases}$$

where $G \setminus e$ is the graph obtained by deleting e
and G/e is the graph obtained by contracting e.

Medial Graph

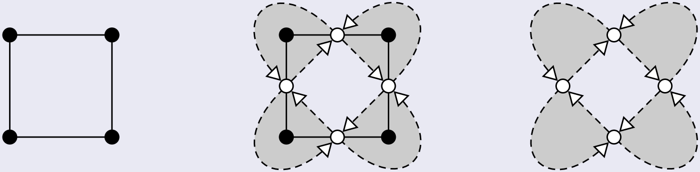
Definition



A plane graph (a), its medial graph (c), and the two graphs superimposed (b).

Directed Medial Graph

Definition

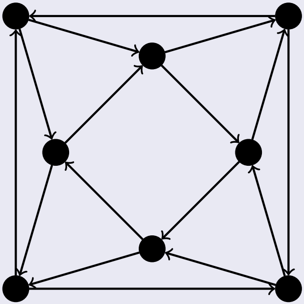


(a) (b) (c)

A plane graph (a), its directed medial graph (c), and the two graphs superimposed (b).

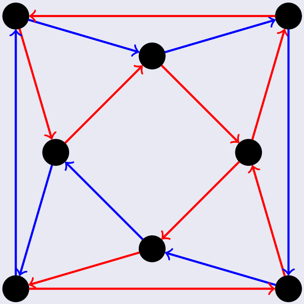
Definition

1. Digraph is Eulerian if "in degree" = "out degree".



Definition

1. Digraph is Eulerian if "in degree" = "out degree".
2. Eulerian partition of an Eulerian digraph $\vec{G}$ is a partition of the edges of $\vec{G}$ such that each part induces an Eulerian digraph.



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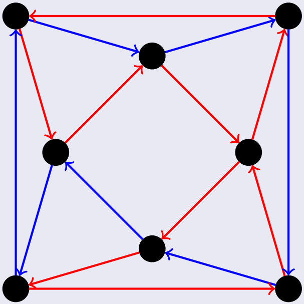
3. Let $\pi_\kappa(\vec{G})$ be the set of Eulerian partitions of $\vec{G}$ into at most κ parts.



$\kappa \geq 2$

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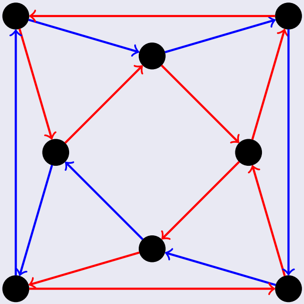
3. Let $\pi_\kappa(\vec{G})$ be the set of Eulerian partitions of $\vec{G}$ into at most κ parts.
4. Let $\mu(c)$ be the number of monochromatic vertices in c.



$$\kappa \geq 2$$
$$\mu(c) = 1$$

Crucial Identity

Theorem (Ellis-Monaghan)

For a plane graph G,

$$\kappa \, \mathrm{T}(G; \kappa + 1, \kappa + 1) = \sum_{c \,\in\, \pi_{\kappa}(\vec{G}_m)} 2^{\mu(c)}.$$

Connection to Holant

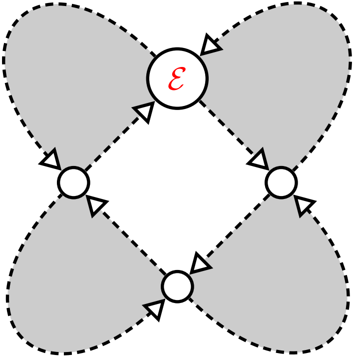
Then

$$\sum_{c \in \pi_\kappa(\vec{G}_m)} 2^{\mu(c)} = \text{Holant}_\kappa(G_m; \langle 2, 1, 0, 1, 0 \rangle),$$

where

$$\mathcal{E}\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \begin{cases} 2 & \text{if } w = x = y = z \\ 1 & \text{if } w = x \neq y = z \\ 0 & \text{if } w = y \neq x = z \\ 1 & \text{if } w = z \neq x = y \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{E} = \langle 2, 1, 0, 1, 0 \rangle$.



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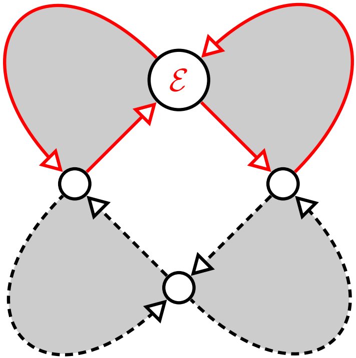

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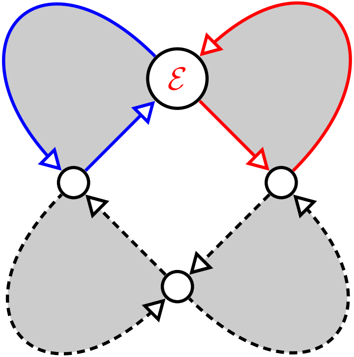

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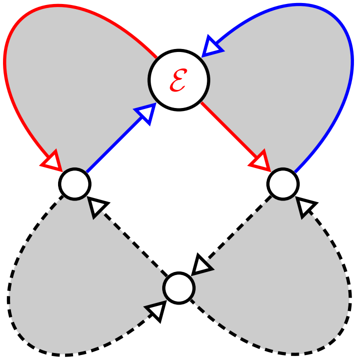

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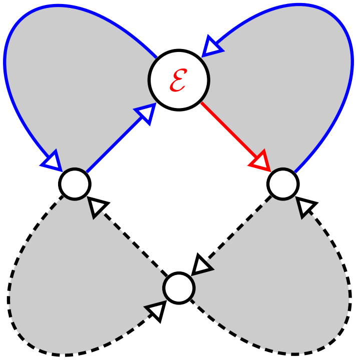


Upshot

Corollary (Cai, Guo, W 14)

For a plane graph G,

$$\kappa T(G; \kappa + 1, \kappa + 1) = \mathrm{Holant}_{\kappa}(G_m; \langle 2, 1, 0, 1, 0 \rangle),$$

Upshot

Corollary (Cai, Guo, W 14)

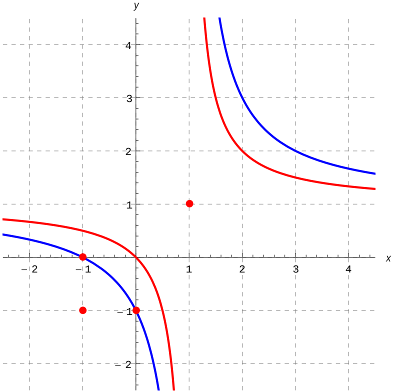
For a plane graph G,

$$\kappa T(G; \kappa+1, \kappa+1) = \text{Holant}_\kappa(G_m; \langle 2,1,0,1,0\rangle),$$

and #**P-hard** *over planar graphs for $\kappa \geq 2$.*

Theorem (Vertigan)

For any $x, y \in \mathbb{C}$, the problem of evaluating the Tutte polynomial at (x, y) over planar graphs is #**P-hard** *unless $(x-1)(y-1) \in \{1, 2\}$ or $(x, y) \in \{(\pm 1, \pm 1), (\omega, \omega^2)\}$, where $\omega^3 = 1$, which are computable in polynomial time.*



Outline

Theorem and Proof Outline

Theorem (Guo, W 13)

Counting Eulerian Orientations is $\#$**P-hard** *over planar 4-regular graphs.*

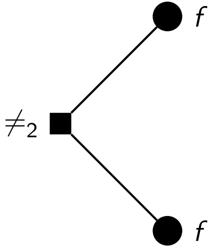
Theorem (Guo, W 13)

*Counting Eulerian Orientations is $\#$**P-hard** over planar 4-regular graphs.*

Proof.

$$\begin{aligned} \text{Pl-Holant}_{2}(\langle 2,1,0,1,0\rangle) &\le_T \quad \vdots \\ &\le_T \#\text{Pl-4Reg-EO} \end{aligned}$$

PI-Holant $(\neq_2 \mid f)$



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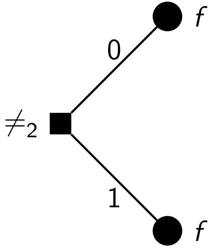

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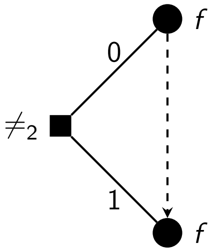

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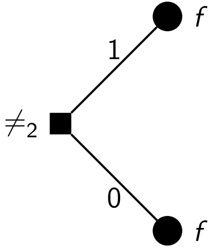

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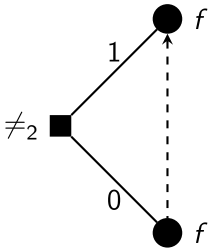

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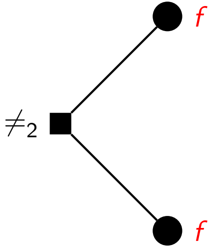
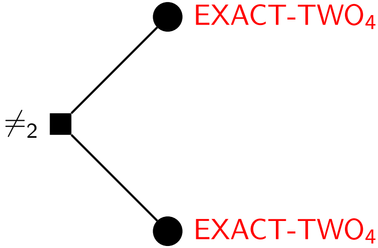


Pl-Holant$(\neq_2 \mid \text{EXACT-TWO}_4)$



Theorem (Guo, W 13)

*Counting Eulerian Orientations is #**P-hard** over planar 4-regular graphs.*

Proof.

$$\begin{aligned}\text{Pl-Holant}_2(\langle 2,1,0,1,0\rangle) &\leq_T \quad \vdots \\ &\leq_T \text{Pl-Holant}(\neq_2 \mid \text{EXACT-TWO}_4) \\ &\equiv_T \#\text{Pl-4Reg-EO}\end{aligned}$$

Matrix Form

Definition

Let f be a function of arity 4 over the Boolean domain with $f(w,x,y,z) = f^{wxyz}$. Then we express f as the matrix

$$\begin{bmatrix} f^{0000} & f^{0010} & f^{0001} & f^{0011} \\ f^{0100} & f^{0110} & f^{0101} & f^{0111} \\ f^{1000} & f^{1010} & f^{1001} & f^{1011} \\ f^{1100} & f^{1110} & f^{1101} & f^{1111} \end{bmatrix}.$$

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Over the Boolean domain, the matrix form of $f = \langle 2, 1, 0, 1, 0 \rangle$ is

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The matrix form of $g = \text{EXACT-TWO}_4$ is

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Theorem and Proof Outline

Theorem (Guo, W 13)

*Counting Eulerian Orientations is #**P-hard** over planar 4-regular graphs.*

Proof.

$$\begin{aligned}
\text{Pl-Holant}_{2}(\langle 2,1,0,1,0\rangle) = &\quad \text{Pl-Holant}\left(\begin{bmatrix} 2&0&0&1\\ 0&1&0&0\\ 0&0&1&0\\ 1&0&0&2 \end{bmatrix}\right) \\
\leq_T &\qquad \vdots \\
\leq_T &\ \text{Pl-Holant}\left(\neq_2 \mid \begin{bmatrix} 0&0&0&1\\ 0&1&1&0\\ 0&1&1&0\\ 1&0&0&0 \end{bmatrix}\right) \\
= &\quad \text{Pl-Holant}(\neq_2 \mid \text{EXACT-TWO}) \\
\equiv_T &\ \#\text{Pl-4Reg-EO}
\end{aligned}$$

Holographic Transformation

Under a holographic transformation by $Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$,

$$\begin{aligned} \text{Pl-Holant}\left(\neq_2 \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) &= \text{Pl-Holant}\left(=_2 \mid \frac{1}{2} \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}\right) \\ &\equiv_T \text{Pl-Holant}\left(\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}\right). \end{aligned}$$

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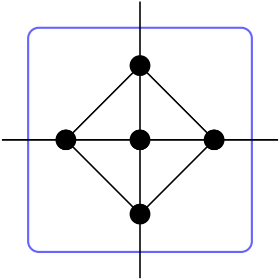
Gadget Construction

Assign the function with matrix $\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$ to every vertex of this gadget...



...to get a function with matrix

$$16 \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}.$$

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&\le_T \text{Pl-Holant}(\tfrac{1}{2}\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}) \\
&\le_T \text{Pl-Holant}(\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}) \\
&= \text{Pl-Holant}(\neq_2 \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}) \\
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Interpolation

Let $T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$.

Interpolation

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Then

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} T^{-1} \quad \text{and} \quad \frac{1}{2} \begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix} T^{-1}.$$

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Given a 4-regular graph G, let

$$p(G; x, y, z) = \text{Holant}(G; T \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} T^{-1}).$$

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Let $T = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$.

Then

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} T^{-1} \quad \text{and} \quad \frac{1}{2}\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix} = T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix} T^{-1}.$$

Given a 4-regular graph G, let

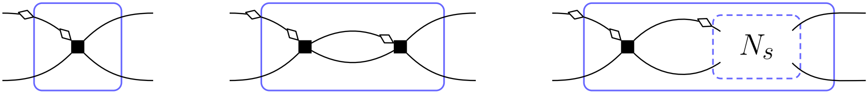
$$p(G; x, y, z) = \text{Holant}\left(G;\ T \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & z \end{bmatrix} T^{-1}\right).$$

Then

$$p(G; 1, 1, 3) = \text{Holant}\left(G;\ T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} T^{-1}\right).$$

Interpolation

Assign the function with matrix $T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix} T^{-1}$ to every vertex of N_s...



N_1 N_2 N_{s+1}

...to get a function with matrix $T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}^s T^{-1}$.

Let G_s be obtained from G by replacing every vertex with N_s.

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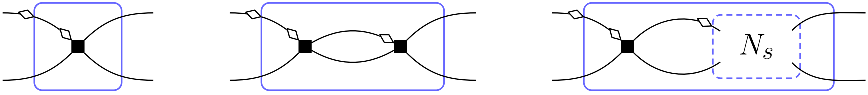

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Let G_s be obtained from G by replacing every vertex with N_s.

Then

$$\begin{aligned} p(G; 1^s, 6^s, 13^s) &= \text{Holant}\left(G; T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}^s T^{-1}\right) \\ &= \text{Holant}\left(G_s; T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix} T^{-1}\right). \end{aligned}$$

Let $c_{jk\ell}$ be the coefficient of $x^j y^k z^\ell$ in $p(x, y, z)$.

Then (with n vertices in G)

$$p(G; 1^s, 6^s, 13^s) = \sum_{j+k+\ell=n} (6^k 13^\ell)^s c_{jk\ell}$$

is a full rank Vandermonde system:

- row index s;
- column index (j, k, ℓ).

QED

Theorem and Proof Outline

Theorem (Guo, W 13)

*Counting Eulerian Orientations is #**P-hard** over planar 4-regular graphs.*

Proof.

$$
\begin{aligned}
\text{Pl-Holant}_{2}(\langle 2,1,0,1,0\rangle) &= \text{Pl-Holant}\left(\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix}\right) \\
&\le_T \text{Pl-Holant}\left(\frac{1}{2}\begin{bmatrix} 19 & 0 & 0 & 7 \\ 0 & 7 & 5 & 0 \\ 0 & 5 & 7 & 0 \\ 7 & 0 & 0 & 19 \end{bmatrix}\right) \\
&\le_T \text{Pl-Holant}\left(\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}\right) \\
&= \text{Pl-Holant}\left(\neq_2 \mid \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\right) \\
&\le_T \text{Pl-Holant}(\neq_2 \mid \text{EXACT-TWO}) \\
&\equiv_T \#\text{Pl-4Reg-EO}
\end{aligned}
$$

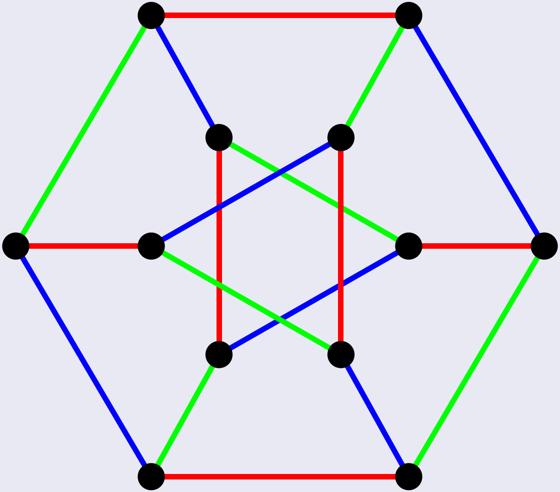
□

Techniques: holographic transformation, gadget construction, interpolation

Outline

Edge Coloring

Definition



Theorem (Cai, Guo, W 14)

*Counting edge colorings with κ colors is #**P-hard** over planar κ-regular graphs for $\kappa \geq 3$.*

Proof.

$$
\begin{aligned}
\text{Pl-Holant}_{\kappa}(\langle 2,1,0,1,0\rangle) &\leq_T \quad \vdots \\
&\leq_T \quad \vdots \\
&\leq_T \#\text{Pl-}\kappa\text{Reg-}\kappa\text{EdgeColoring}
\end{aligned}
$$

Theorem and Proof Outline

Theorem (Cai, Guo, W 14)

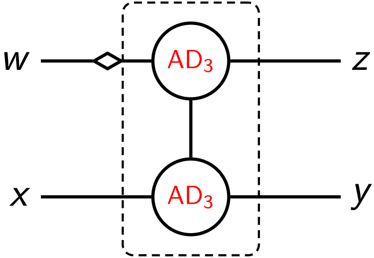
*Counting edge colorings with κ colors is #**P-hard** over planar κ-regular graphs for $\kappa \geq 3$.*

Proof.

$$
\begin{aligned}
\text{Pl-Holant}_\kappa(\langle 2, 1, 0, 1, 0 \rangle) &\leq_T \quad \vdots \\
&\leq_T \text{Pl-Holant}_\kappa(\text{ALL-DISTINCT}_\kappa) \\
&= \#\text{Pl-}\kappa\text{Reg-}\kappa\text{EdgeColoring}
\end{aligned}
$$

Let $\mathrm{AD}_3 = \text{ALL-DISTINCT}_3$.

$$\mathrm{Holant}_3(G; \langle 0,1,1,0,0 \rangle) = \mathrm{Holant}_3(G'; \mathrm{AD}_3)$$



$$f\left(\begin{smallmatrix} w & z \\ x & y \end{smallmatrix}\right) = \langle 0,1,1,0,0 \rangle = \begin{cases} 0 & \text{if } w = x = y = z \\ 1 & \text{if } w = x \neq y = z \\ 1 & \text{if } w = y \neq x = z \\ 0 & \text{if } w = z \neq x = y \\ 0 & \text{otherwise.} \end{cases}$$

Gadget Construction

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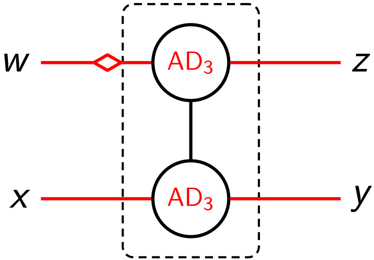

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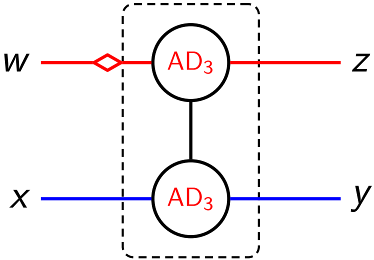

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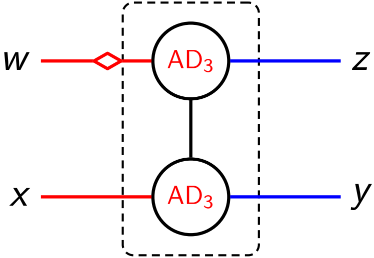

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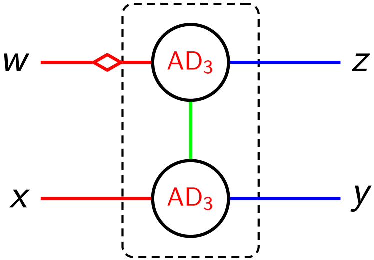

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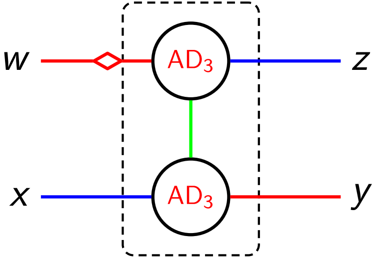

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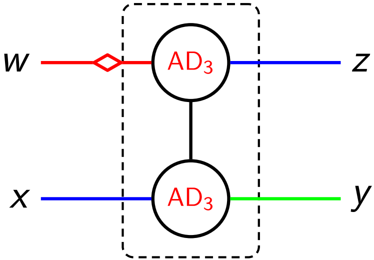

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Theorem (Cai, Guo, W 14)

Counting edge colorings with κ colors is $\#$**P-hard** *over planar κ-regular graphs for $\kappa \geq 3$.*

Proof.

$$\begin{aligned} \text{Pl-Holant}_\kappa(\langle 2,1,0,1,0\rangle) &\leq_T \text{Pl-Holant}_\kappa(\langle 0,1,1,0,0\rangle) \\ &\leq_T \text{Pl-Holant}_\kappa(\text{ALL-DISTINCT}_\kappa) \\ &= \#\text{Pl-}\kappa\text{Reg-}\kappa\text{EdgeColoring} \end{aligned}$$

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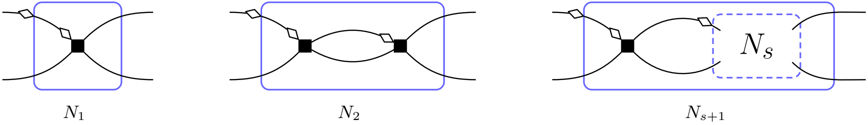
\end{aligned}$$

Assign $\langle 0, 1, 1, 0, 0 \rangle$ to every vertex of N_s...



...to get a function f_s.

Then $f_s = M^s f_0$, where

$$M = \begin{bmatrix} 0 & \kappa - 1 & 0 & 0 & 0 \\ 1 & \kappa - 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad f_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Let G_s be obtained from G by replacing every vertex with N_s.

Spectral decomposition $M = P\Lambda P^{-1}$, where

$$P = \begin{bmatrix} 1 & 1-\kappa & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} \kappa-1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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Let

$$f(x) = P \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1} f_0 = \begin{bmatrix} \frac{x-1}{\kappa} + 1 \\ \frac{x-1}{\kappa} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

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Given a 4-regular graph G, let $p(G; x) = \mathrm{Holant}_\kappa(G; f(x))$.

Polynomial Interpolation

Spectral decomposition $M = P \Lambda P^{-1}$, where

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$$f(x) = P \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} P^{-1} f_0 = \begin{bmatrix} \frac{x-1}{\kappa} + 1 \\ \frac{x-1}{\kappa} \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Given a 4-regular graph G, let $p(G; x) = \operatorname{Holant}_\kappa(G; f(x))$.
Then $p(G; \kappa + 1) = \operatorname{Holant}_\kappa(G; \langle 2, 1, 0, 1, 0 \rangle)$
and $p\left(G; (\kappa - 1)^{2s}\right) = \operatorname{Holant}_\kappa(G; f_{2s}) = \operatorname{Holant}_\kappa(G_{2s}; \langle 0, 1, 1, 0, 0 \rangle)$.

Polynomial Interpolation

If G has n vertices, then $p(G, x)$ has degree n.

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Since $(\kappa - 1)^{2s}$ is distinct for $0 \leq s \leq n$, we can efficiently compute the coefficients of $p(G, x)$.

QED

Theorem and Proof Outline

Theorem (Cai, Guo, W 14)

*Counting edge colorings with κ colors is #**P-hard** over planar κ-regular graphs for $\kappa \geq 3$.*

Proof.

$$\begin{aligned}\text{Pl-Holant}_\kappa(\langle 2,1,0,1,0\rangle) &\leq_T \text{Pl-Holant}_\kappa(\langle 0,1,1,0,0\rangle) \\ &\leq_T \text{Pl-Holant}_\kappa(\text{ALL-DISTINCT}_\kappa) \\ &= \#\text{Pl-}\kappa\text{Reg-}\kappa\text{EdgeColoring}\end{aligned}$$

$\square$

Outline

Dichotomy Theorems

Theorem (Cai, Kowalczyk, W 12)

Z(*binary*) *over planar 3-regular directed graphs.*

Theorem (Guo, W 13)

$\#$CSP(*symmetric set*) *over planar graphs.*

Theorem (Cai, Guo, W 13)

Holant(*symmetric set*) *over general graphs.*

Theorem (Cai, Guo, W 14)

Holant_{κ}(*symmetric domain invariant*) *over planar 3-regular graphs for* $\kappa \geq 3$.

Theorem (Guo, W 13)

Counting Eulerian Orientations is $\#$**P-hard** *over planar 4-regular graphs.*

Theorem (Cai, Guo, W 14)

Counting edge colorings with κ *colors is* $\#$**P-hard** *over planar* κ*-regular graphs for* $\kappa \geq 3$.

Theorem (Guo, W 13)

*Counting Eulerian Orientations is $\#$**P-hard** over planar 4-regular graphs.*

Theorem (Cai, Guo, W 14)

*Counting edge colorings with κ colors is $\#$**P-hard** over planar r-regular graphs for $\kappa \geq r \geq 3$.*

Thank You