Overview

- Random Variables are used to model, e.g.
  - unpredictable interarrival times, service times
  - unpredictable routing of client requests
  - unpredictable system residence times, etc.

- formalize intuitive concepts, such as
  - skewed distributions, bursty arrivals
  - heavy-tailed service times
  - average value
Random Variable: Definition

A random variable, $X$, is a variable that takes on a value as a result of an experiment.

e.g.,

- $N = \# \text{ processors requested by random O2K job}$
  - $N \in \{1, 2, \ldots, 256\}$
- $R = \text{ runtime of a randomly selected job}$
  - $R \in \Re^+$
- $X = \text{ status of the server when a job arrives}$
  - $X \in \{0, 1\} \quad 0 = \text{ idle}, \ 1 = \text{ busy}$

can have functions of these variables: $X^2$, $2X$, $W+X$
Probability Mass Function (pmf): $p_X(k)$

- $p_X(k) \equiv P[X = k]$  
  - $X$: name of the r.v.  
  - $k$: name of possible value of the r.v.

- N = # processors requested by a random O2K job
  - $N \in \{1, 2, \ldots, 256\}$
  - $p_N(16) =$ probability that the number requested, $N$, is 16

- $X \sim \text{discrete uniform}(a, b)$
  - $p_X(k) \equiv P[X = k]$  
  - $p_X(k) = \frac{1}{(b-a)+1}$, $a, b \in \{-2,-1,0,1,2,\ldots\}$
  - $a \leq k \leq b$

Example models:
- $X =$ id of the processor that will receive a message ($a = 1, b = 256$)
- $X =$ id of the disk that will service an I/O request ($a = 1, b = 8$)
\( X \sim \text{Zipf}(\alpha) \)

\[
p_X(k) = \frac{1/k^{1+\alpha}}{\sum_{k=1}^{\infty} 1/k^{1+\alpha}}, \quad \alpha > 0, \quad k = 1, 2, 3, ... \]

e.g., \( X \sim \text{Zipf}(0) \)

Example model: \( X = \) the video or file requested by a client

highly skewed, long tailed

\[ p_X(x) = P[X = x] \]
e.g.,

\( R = \) runtime of a random O2K job, \( R \in [0, 401 \text{ hrs}] \)

\[
p_R(3.002 \text{ hr}) = 0 \]

\[
p_R(x) = 0, \quad 0 \leq x \leq 401 \text{ hr} \]

\( p_X(x) \) is not a useful characterization of a continuous r.v.
pmf, $p_X(x)$: Properties

$p_X(x) = P[X = x]$

e.g.,

$N = \#$ processors requested by a random O2K job

$R = \text{runtime of a random O2K job, } R \in [0, 401]$  

Properties of a pmf:

- $0 \leq p_X(x) \leq 1$  
  A function $p(x)$ is a valid pmf iff it satisfies these properties

- $\sum_{x_i \mid p_X(x_i) > 0} p_X(x_i) = 1$

$$\sum_{x_i \mid p_X(x_i) > 0} p_X(x_i) = \begin{cases} 0 & \Rightarrow X \text{ is a continuous RV} \\ (0,1) & \Rightarrow X \text{ is a mixed RV} \\ 1 & \Rightarrow X \text{ is a discrete RV} \end{cases}$$

Cumulative Distribution Fn. (CDF): $F_X(x)$

$F_X(x) = P[X \leq x]$

Properties:

- $P[a \leq X \leq b] = F(a) - F(b)$

- $P[X > a] = 1 - F(a)$

- $a < b \Rightarrow F_X(a) < F_X(b)$

- $\lim_{x \to -\infty} F_X(x) = 0$, $\lim_{x \to \infty} F_X(x) = 1$

- e.g.,

- $R = \text{runtime of a random O2K job, } R \in [0, 401 \text{ hrs}]$

- $P[1 \text{ h} \leq X \leq 10 \text{ h}] = F(10) - F(1) \approx 0.2$
X ~ uniform(a, b)

\[ F_X(x) = \begin{cases} \frac{x - a}{b - a} & \text{for } a \leq x \leq b \\ 0 & \text{for } x < a \\ 1 & \text{for } x > b \end{cases} \]

\[ \lim_{x \to a^+} F_X(x) = 0, \quad \lim_{x \to b^-} F_X(x) = 1 \]

Example Model:
\( X = \text{rotational latency component of a disk access time} \)

CDF, \( F_X(x) \): discrete RVs

\[ F_X(x) = \begin{cases} \frac{x - a + 1}{b - a} & \text{for } a \leq x \leq b \\ 0 & \text{for } x < a \\ 1 & \text{for } x > b \end{cases} \]

\[ \lim_{x \to a^+} F_X(x) = 0, \quad \lim_{x \to b^-} F_X(x) = 1 \]

Example Model:
\( X = \text{id of the processor that will receive a message (a=1, b=256)} \)
$X \sim \text{exponential}(\lambda)$

$F_X(x) = P[X \leq x]$

\[ \lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to \infty} F_X(x) = 1 \]

e.g.,

$X \sim \text{exponential}(\lambda), \quad \lambda > 0$

$F_X(x) = 1 - e^{-\lambda x}$

$x \geq 0$

$P[10 \leq X \leq 30] = F(30) - F(10) = 0.38$

Example Model:

$X =$ time between consecutive client arrivals to a media server

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**Expected Value, $E[X]$**

Consider the average value computed from a set of measurements of $X$, and let

- $m =$ # measurements, $(y_1, y_2, \ldots, y_m) =$ measured values
- $n_i =$ # times $X = x_i$ was observed

then, $\bar{X} = \frac{1}{m} \sum_{j=1}^{m} y_j = \frac{1}{m} \sum_{x_i} n_i x_i = \sum_{x_i} \frac{n_i}{m} x_i$

\[ E[X] = \sum_{x_i \in \mathbb{T}} x_i p_X(x_i) \quad \text{if } X \text{ is discrete} \]

$X \sim \text{uniform}(a,b)$:

\[ E[X] = \sum_{i=a}^{b} \frac{i}{b-a+1} = \sum_{i=a}^{b} \frac{i}{b-a+1} \left( \frac{i^2 - (i-1)^2}{i^2} \right) = \frac{b}{2} \]

$E[X] = ? \quad \text{if } X \text{ is continuous}$
\(X \sim \text{Bernoulli}(p)\)

\[X = \mathbb{E}[X] = \sum_{x \in \mathbb{F}} x \cdot p_X(x)\] if \(X\) is discrete

e.g., \(X \sim \text{Bernoulli}(p), \quad 0 \leq p \leq 1\)

- \(x \in \{0, 1\} = \text{probability of success on a single trial}\)
- \(p_X(0) = 1 - p, \quad p_X(1) = p\)
- \(\mathbb{E}[X] = 0 \times (1 - p) + 1 \times p = p\)

Example model: \(X = \text{status of the server at a random time}\)
\(X \in \{0, 1\}, \quad \text{where } 0 \equiv \text{idle, } 1 \equiv \text{busy}\)
\(X \sim \text{Bernoulli}(U), \quad \mathbb{E}[X] = U\)

**Summary**

- \(X = \text{variable that takes on a value as a result of an experiment}\)
- \(p_X(x) = P[X = x]\)
- \(F_X(x) = P[X \leq x]\)
- \(P[x \leq X \leq y] = F(y) - F(x)\)
- \(F_X(x)\) is a complete characterization of \(X\)
- if \(X\) is discrete,
- \(p_X(x)\) is a complete characterization of \(X\)
- \(\bar{X} \equiv \mathbb{E}[X] = \sum_{x_i \in \mathbb{T}} x_i \cdot p_X(x_i)\) if \(X\) is discrete
Summary

$X \sim \text{discrete uniform}(a, b): \quad E[X] = \frac{b + a}{2}$

$X \sim \text{Zipf}(\alpha): \quad p_X(k) = \frac{1/k^{\alpha}}{\sum_{k=1}^{\infty} 1/k^{\alpha}}, \quad \alpha > 0, \quad k = 1, 2, 3, ...$

$X \sim \text{exponential}(\lambda): \quad F_X(x) = 1 - e^{-\lambda x}, \quad \lambda > 0, \quad x \geq 0$

$X \sim \text{Bernoulli}(p): \quad p_X(0) = 1 - p, \quad p_X(1) = p$

Questions?