NETWORKS OF WAITING LINES*

JAMES R. JACKSON†

University of California, Los Angeles, California

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ONE OF THE least understood classes of operations problems is that concerned with the design, loading, and, especially, the scheduling of discrete, statistically varying flows through complex networks. The present paper abstracts what is perhaps the simplest theoretical question related to this class of problems, and derives expressions for certain steady-state parameters. To put this theoretical question in context, let us give it a hypothetical referent.

'A machine shop' has several departments each containing a fixed number of identical machines. Each department is a multiserver system of the usual type (the waiting jobs are pooled in a single line, a given job is definitely assigned to a fixed machine when its turn comes up, and service times are exponentially distributed). However, arrivals at a given department come both from other departments in the shop and from outside the shop. Those coming to any department from outside arrive in a Poisson-type time series. The flow pattern of jobs inside the shop is most easily described by saying that when a given department finishes a job, that job either goes to some specified department or out of the system, its particular course being governed by a fixed probability distribution associated with the particular department that it is leaving.

If mean arrival rates at the various departments are properly defined, then the result is a steady-state distribution in which the waiting-line lengths of the departments are independent, and are exactly like those of the 'ordinary' multiserver systems that they resemble.

This paper is a part of an extended study of problems arising in machine-shop operations, carried on under a contract with the Office of Naval Research. Further details and related problems are treated in references 2, 4, 5, and 6.

Theoretical background

Erlang and others have treated the following elementary steady-state problem in waiting-line theory.[1, 3] Customers arrive in a Poisson-type

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time series at mean rate λ. They are handled on a first-come, first-serve basis by a system of n identical servers, the servicing times being exponentially distributed with mean $1/\mu$. If k denotes the number of customers waiting and in service; it is shown that if $\lambda<\mu n$, then a steady-state distribution of k is given by

$$P_k=\begin{cases} P_0\,(\lambda/\mu)^k/k!, & (k=0,\,1,\,\cdots,\,n) \\ P_0\,(\lambda/\mu)^k/n!\,n^{k-n}, & (k=n,\,n+1,\,\cdots) \end{cases}$$

The number P_0 can of course be determined from the equation $\sum P_k=1$.

Results

The present paper deals with the situation in which there are M 'departments,' each being a system like that described above, and with number of servers, mean arrival rate, and mean holding time varying from department to department. Specifically, let there be Departments 1, 2, $\cdots$, M. For $m=1,\,2,\,\cdots,\,M$:

1. Department m contains n_m servers.
2. Customers *from outside the system* arrive in Department m in a Poisson-type time series at mean rate λ_m (customers will also arrive at this department from other departments).
3. Customers arriving in Department m (from inside or outside the system) are served in turn. The serving time is exponentially distributed with mean $1/\mu_m$, a given customer being assigned once and for all to a fixed server when his turn comes up.
4. Once served in Department m, a customer goes (instantaneously) to Department k ($k=1,\,2,\,\cdots,\,M$) with probability θ_{km}; his total service is completed with probability $1-\sum_k \theta_{km}$.

Assumption (4) is the basis for calling this system a 'network' of waiting lines.

For $m=1,\,2,\,\cdots,\,M$, let Γ_m be the average arrival rate of customers at Department m from any source, inside or outside the system. It is easily seen that in a steady state, we must have

$$\Gamma_m=\lambda_m+\sum\nolimits_k \theta_{mk}\,\Gamma_k.$$

The Γ_m of the present problem plays the same role as the λ of the elementary problem described previously.

Now let k_m denote the number of customers waiting and in service at Department m ($m=1,\,2,\,\cdots,\,M$); and define the 'state of the system' (actually a function of time) as the vector $(k_1,\,k_2,\,\cdots,\,k_M)$. Then the following theorem is true:

THEOREM. *Define* $P_k{}^m$ $(m=1, 2, \cdots, M, k=0, 1, 2, \cdots)$ *by the following equations (where the* $P_0{}^m$ *are determined by conditions* $\sum_k P_k{}^m=1$):

$$P_k{}^m=\begin{cases} P_0{}^m\ (\Gamma_m/\mu_m)^k/k!, & (k=0,\ 1,\ \cdots,\ n_m) \\ P_0{}^m\ (\Gamma_m/\mu_m)^k/n_m!\ (n_m)^{k-n_m}. & (k=n_m,\ n_{m+1},\ \cdots) \end{cases}$$

A steady-state distribution of the state of the above-described system is given by the products

$$P(k_1,\ k_2,\ \cdots,\ k_M)=P_{k_1}{}^1\ P_{k_2}{}^2\ \cdots\ P_{k_M}{}^M,$$

provided $\Gamma_m<\mu_m\, n_m$ *for* $m=1, 2, \cdots, M$.

This theorem says, in essence, that at least so far as steady states are concerned, the system with which we are concerned behaves *as if* its departments were such independent elementary systems as are discussed above. This conclusion is far from surprising in view of recent papers by E. J. BURKE[2] and E. REICH.[5]

PROOF. The last condition of the theorem guarantees that $\sum_k P_k{}^m$ will converge, and is evidently a necessary condition for the existence of a steady state. To establish that the given distribution then defines a steady state, we follow the general approach used by FELLER. Let $P_{k_1,\ \cdots,\ k_M}(t)$ be the probability of state $(k_1,\ k_2,\ \cdots,\ k_M)$ at time t. From a straightforward consideration of the ways in which the system can reach state $(k_1,\ k_2,\ \cdots,\ k_M)$, it turns out that

$$\begin{aligned} P_{k_1,\ \cdots,\ k_M}(t+h)=&\{1-(\textstyle\sum \lambda_i)\ h-[\sum \alpha_i(k_i)\ \mu_i]\ h\}\ P_{k,\cdots,\ k_M}(t) \\ &+\textstyle\sum \alpha_i(k_i+1)\ \mu_i\ \theta_i{}^*\ h\ P_{k_1,\cdots,\ k_i+1,\ \cdots,\ k_M}(t) \\ &+\textstyle\sum \lambda_i\ \delta_i\ h\ P_{k_1,\ \cdots,\ k_i-1,\ \cdots,\ k_M}(t) \\ &+\textstyle\sum\sum \alpha_j(k_j+1)\ \mu_j\ \theta_{ij}\ h\ P_{k_1,\cdots,\ k_j+1,\ \cdots,\ k_i-1,\ \cdots,\ k_m}(t)+o(h); \end{aligned}$$

where

$$\theta_i{}^*=1-\textstyle\sum_k \theta_{ki},$$

$$\alpha_i(k)=\min\{k,\ n_i\},$$

$$\delta_i=\min\{k_i,\ 1\}.$$

Following the usual process of transferring the $P_{k_1,\ k_2,\ \cdots,\ k_M}(t)$ from right to left, dividing by h, and taking the limit as h approaches zero, one obtains a set of differential equations (which will not be written out here). To prove that the given distribution is a steady-state solution of these equations, it is enough to show that the derivatives in these equations are all made zero by setting $P_{k_1,\ k_2,\ \cdots,\ k_M}(t)$ equal to $P(k_1,\ \cdots,\ k_M)$; that is, to show that:

$$[\sum \lambda_i + \sum \alpha_i(k_i)\,\mu_i]\,P(k_1, \cdots, k_M) = \sum \alpha_i(k_i+1)\,\mu_i\,\theta_i^*$$
$$P(k_1, \cdots, k_i+1, \cdots, k_M) + \sum \lambda_i \delta_i\, P(k_1, \cdots, k_i-1, \cdots, k_M)$$
$$+ \sum \sum \alpha_j(k_j+1)\,\mu_j\,\theta_{ij}\,P(k_1, \cdots, k_j+1, \cdots, k_i-1, \cdots, k_M).$$

Now the following relations are easily seen from the equations defining the $P(k_1, \cdots, k_M)$ and the P_k^m:

$$\frac{P(k_1, \cdots, k_i+1, \cdots, k_M)}{P(k_1, \cdots, k_i, \cdots, k_M)} = \frac{\Gamma_i}{\mu_i\,\alpha_i(k_i+1)},$$

$$\frac{P(k_1, \cdots, k_i-1, \cdots, k_M)}{P(k_1, \cdots, k_i, \cdots, k_M)} = \frac{\mu_i\,\alpha_i(k_i)}{\Gamma_i},$$

$$\frac{P(k_1, \cdots, k_j+1, \cdots, k_i-1, \cdots, k_M)}{P(k_1, \cdots, k_j, \cdots, k_i, \cdots, k_M)} = \frac{\Gamma_j\,\mu_i\,\alpha_i(k_i)}{\mu_j\,\alpha_j(k_j+1)\,\Gamma_i}.$$

Dividing the equation to be established by $P(k_1, \cdots, k_M)$, substituting from these formulas, and noting that $\delta_i\,\alpha_i(k_i) = \alpha_i(k_i)$, it remains to show that:

$$\sum \lambda_i + \sum \alpha_i(k_i)\,\mu_i = \sum \theta_i^*\,\Gamma_i + \sum [\lambda_i\,\mu_i\,\alpha_i(k_i)/\Gamma_i]$$
$$+ \sum \sum [\mu_i\,\alpha_i(k_i)/\Gamma_i]\,\theta_{ij}\,\Gamma_j.$$

But
$$\sum \theta_i^*\,\Gamma_i = \sum_i (1 - \sum_k \theta_{ki})\,\Gamma_i = \sum \lambda_i,$$

from the defining equation of the Γ_i. Also from the same equation,

$$\sum \sum [\mu_i\,\alpha_i(k_i)/\Gamma_i]\,\theta_{ij}\,\Gamma_j = \sum_i [\mu_i\,\alpha_i(k_i)/\Gamma_i] \sum_j \theta_{ij}\,\Gamma_j,$$
$$= \sum_i [\mu_i\,\alpha_i(k_i)/\Gamma_i]\,(\Gamma_i - \lambda_i),$$
$$= \sum \mu_i\,\alpha_i(k_i) - \sum \lambda_i\,\mu_i\,\alpha_i(k_i)/\Gamma_i.$$

Substituting accordingly, the necessary equality is established.

REFERENCES

1. E. Brockmeyer, H. L. Halstrom, and Arne Jensen, "The Life and Works of A. K. Erlang," *Transactions of the Danish Academy of Technical Sciences*, No. 2, 1948.
2. P. J. Burke, "The Output of a Queuing System," *Opns. Res.* **4**, 699 (1956).
3. William Feller, "An Introduction to Probability Theory and Its Applications," Wiley, 1950 (*see especially* pp. 378–79).
4. G. C. Hunt, "Sequential Arrays of Waiting Lines," *Opns. Res.* **4**, 674 (1956).
5. Edgar Reich, "Birth-Death Processes and Tandem Queues," Rand Corporation P-863, June 22, 1956.
6. A. J. Rowe and J. R. Jackson, "Research Problems in Production Scheduling," *J. Indust. Eng.* **7** (May–June, 1956).