GRID-GRAPH PARTITIONING

By

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Abstract

Previous researchers showed that striping techniques produced very good (and, in some cases, asymptotically optimal) partitions when applied to grid graphs. These striping algorithms can be thought of as two-phase methods. The first phase consists of breaking the original problem into smaller, but similar, problems (striping). The second phase (stripe assignment) consists of the actual assignment of cells within the stripes. Results from this reseach show how to improve both phases.

We improve the stripe-assignment phase of Christou, Meyer and Yackel so as to guarantee locally optimal solutions for rectangular grid graphs. It is shown that under certain assumptions, the assignment algorithm of Christou-Meyer will produce a locally optimal solution. This algorithm is extended to handle a larger class of grid graphs. A third algorithm is described that produces a locally optimal solution for the same class of problems and also potentially reduces the chance of producing solutions with certain undesirable characteristics.

In the striping phase, the methods of Yackel-Meyer, Christou-Meyer, and Martin differ in the stripe-height selection process. Yackel-Meyer and Christou-Meyer use genetic algorithms for generating feasible solutions for a general class of grid graphs. Martin considers rectangular domains and transforms the original problem into a knapsack problem and considers a large set of stripe heights. Under the assumption that a stripe-assignment approach satisfying certain generic conditions is given, we derive a dynamic-programming method that generates the best possible set of stripe heights.

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Contents

Chapter 1

Introduction

1.1 Problem Description

Given a graph G = (V,E) and a number of components P, the graph partitioning problem (with uniform node and edge weights) requires dividing the vertices into P groups of equal size such that the number of edges (cut edges) connecting vertices in different groups is minimized. This problem is known to be NP-Complete [GJ79]. In this dissertation, a restricted class of graphs is studied, namely grid graphs (Yackel and Meyer, [Yac93], showed that for this restricted set of graphs the partitioning problem is NP-Hard.). Figure 1 contains an example of a grid graph. The vertices lie at lattice points of a rectangular grid and are connected only to points adjacent on the lattice. Graph partitioning of large uniform grid graphs arises in the context of parallel computation for a variety of problem classes including the solution of PDEs using finite difference schemes [Str89], computer vision [Sch89], and database applications [GMSJ93].

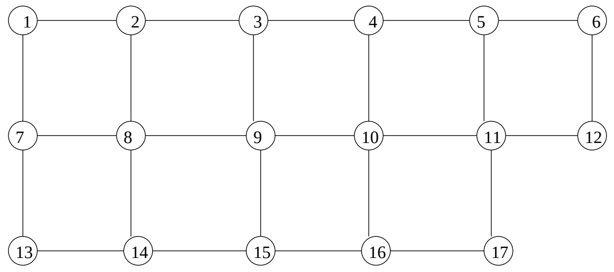


Figure 1: A grid graph

In applications the nodes represent tasks that are to be allocated in a balanced manner among P processors, and the edges represent communication requirements, so that cut edges measure interprocessor communication.

1.2 Alternative Formulations of Graph Partitioning

1.2.1 Quadratic and Mixed Integer Programming Formulations

There are several ways of formulating this graph-partitioning problem. One way is to treat it as a network assignment problem. The vertices will be partitioned into P groups containing specified numbers A_p (number of vertices assigned to processor p), p = 1,...P, of vertices. The corresponding network problem would have P supply nodes with capacities A_p, V demand nodes (note: V is used to denote |V| as well as the set V), each with a demand of 1, and a quadratic objective function (see [Chr96]).

Let,

x_i^p = 1 if vertex i is assigned to processor p

= 0 otherwise,

and consider the following problem:

$$\min \textstyle\sum_{i,j} \left(\sum_{p,p'=1,p\neq p'}^{P} c_{ij}\ x_i^p\ x_j^{p'}\right)$$

$$s.t. \begin{cases} \sum_{i\in V} x_i^p = A_p & p = 1,...,P \\ \sum_{p=1}^{P} x_i^p = 1 & i \in V \\ x_i^p \in \{0,1\} & p = 1,...,P, i \in V \end{cases}$$

$$c_{ij} = \begin{cases} 1 & if (i,j) \in E \\ 0 & otherwise \end{cases}$$

Display 1 : A quadratic-assignment-problem formulation

For the objective function for the QAP formulation, vertices that share an edge are the only combinations that can contribute to the objective function. In this case c_{ij} equals 1. If vertices i and j share an edge and are assigned to the same processor p, then all terms $x_i^p\ x_j^{p'}$ equal zero. If vertex i is assigned to processor p and vertex j is assigned to processor p' with p $\neq$ p', then the $c_{ij}\ x_i^p\ x_j^{p'}$ term counts the cut edge by adding one to the objective value.

The problem can also be formulated as a linear mixed integer programming problem [NW85], by adding additional variables and constraints. In the following formulation [Mey99], the number of edges connecting vertices assigned to the same group (internal edges) is maximized. Define I to be the set of pairs of vertices that are connected by an edge. The formulation becomes:

$$\max \textstyle\sum_{(i,j)\in I,\ \ p\in P} z_{i,j}^p$$

$$s.t. \begin{cases} \sum_{i\in V} x_i^p = A_p & p = 1, ..., P \\ \sum_{p=1}^P x_i^p = 1 & i \in V \\ x_i^p \in \{0,1\} & i \in V, p = 1, ..., P \\ z_{i,j}^p \leq x_i^p & (i,j) \in I, i \in V, p = 1, ..., P \\ z_{i,j}^p \leq x_j^p & (i,j) \in I, i \in V, p = 1, ..., P \end{cases}$$

Display 2: A linear mixed-integer-programming formulation

This research focuses on the equi-partition case in which each group is assigned the same number of vertices (i.e., for p and $p' \in \{1...\ \text{P}\}$ $A_p = A_{p'}$). However, we also discuss extensions of the methods that we develop to the case $|A_p$ - $A_{p'}| \leq 1$ that corresponds to "balancing" partition sizes as much as possible when perfect balance is not possible because P does not divide |V|.

1.2.2 Minimum Perimeter Formulation

Christou and Meyer [CM96] consider a geometric way of formulating this problem, which is useful in terms of generating lower bounds on the optimal value. Figure 2 gives an example of how the original graph is transformed into a domain of cells. Each vertex in the graph becomes a cell (of unit area) and each edge becomes a boundary edge between two cells (domain boundary edges are added as needed to provide four edges for each cell). This geometric view is also the most natural representation of the PDE, vision, and DB applications.

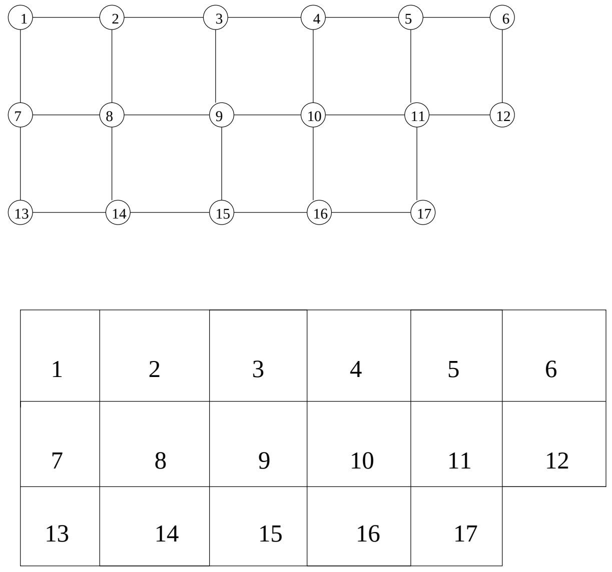


Figure 2: The original graph and the geometric problem.

For the transformed problem, instead of counting cut edges, the sum of the perimeters of the associated components is minimized. Formally, the problem to be solved is:

minimize $\sum_i$ Perim(C_i)

s.t.

each cell is assigned to one processor, and

each processor is assigned an equal number of cells, where $\text{Perim}(C_i)$ equals the perimeter for component C_i.

The relationship between cut edges and the total perimeter is:

$$\text{cut edges} = (\text{total perimeter - perimeter of the boundary of the domain})/2$$

Since the perimeter of the boundary is a constant, minimizing perimeter is equivalent to minimizing cut edges. Figure 3 shows an example of this relationship.(It also illustrates a case in which the component sizes differ by one, because the number of components (six, in this case) does not divide the number of cells (17). Thus, five components are of size three and one component is of size two.)

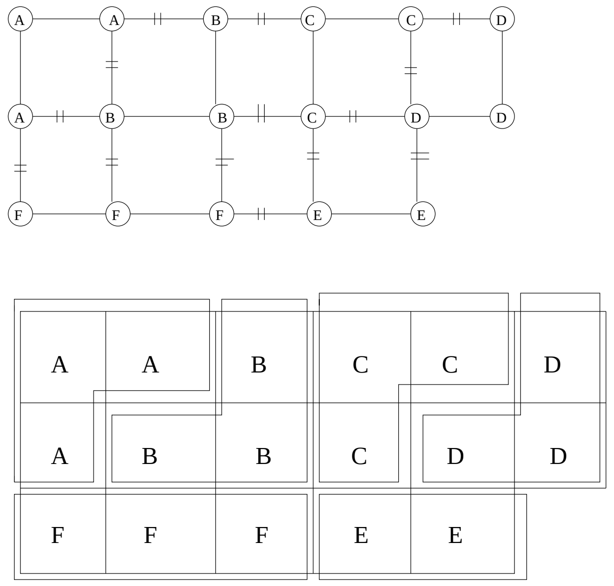


Figure 3: The top figure shows the original graph partitioned. The bottom figure shows an equivalent partition of cells.

In this example the minimum number of cut edges equals 14. The corresponding sum of the perimeters for the components is 46 which is in agreement with the relation above, since the perimeter for the original domain is 18, yielding the number of cut edges as (46-18)/2. The bottom figure is also an example of a stripe-based solution (to be defined formally below; informally this means that components are confined to horizontal bands except possibly for "overflows" at the ends of the bands (overflows do not occur in figure 3)). The perimeter is optimal in this problem instance since it is easily shown to match the lower bound as derived by Yackel, et.al. [YMC97]. (Yackel, et. al. showed that a lower bound for the minimum perimeter when partitioning A cells evenly among P processors is:

$$2P(\lceil 2(A/P)^{0.5} \rceil)$$

All the developed methodology below is based on the geometric model.

1.3 Motivation and Background

This research is a continuation of work done by Yackel and Meyer (YM, [Yac93]), Christou and Meyer (CM, [Chr96]), and Martin [Mar98]. For the partitioning of a grid graph, the methodologies of Christou and Martin [Chr96], [Mar98] both outperform more well-known algorithms such as recursive spectral bisection and geometric mesh partition (both algorithms will be described in Chapter 2)..

Table 1 contains results using the Christou-Meyer methodology [Chr96] and shows how the Distributed Genetic Algorithm (DGA) [Chr96] (this algorithm is described in Chapter 2) outperforms recursive spectral bisection (RSB), implemented using the Chaco package [HL95a], and the Geometric Mesh Partition [GMT95] [MTTV93] for rectangular grids. In tables 1, 2 and 3, all times are measured in seconds and "Gap %" represents the difference between the perimeter generated by the given methodology and the Yackel-Meyer lower bound. (For the 32x31 to be partitioned among 256 components, the Geometric Mesh Partition algorithm produced a negative gap, as marked with an asterisk(*). This indicates that the solution produced was not feasible, i.e., there exists

Problem		RSB		GEOMETRIC		DGA	
M x N	P	Time	Gap %	Time	Gap %	Time	Gap %
32 x 31	8	1.8	6.52	43.6	5.43	6.9	1.08
32 x 31	256	4.3	6.73	152.3	-2.73*	7.1	0.00
32 x 30	64	3.0	6.25	90.4	6.25	7.8	0.00
100 x 100	8	9.0	9.33	111.0	7.39	17.7	2.28
128 x 128	128	85.5	14.13	539.9	7.13	15.5	1.63
256 x 256	256	227.8	13.25	3304.2	4.15	36.9	0.00
512 x 512	512	-	-	-	-	123.8	0.56

Table 1: Christou-Meyer versus other graph-partitioning algorithms for rectangular grids

at least one component that was assigned at least two cells more than at least one other component.) For more information about the chosen set of problems see [Chr96].

Martin [Mar98] also studied rectangular grids and obtained the results contained in table 2, via a knapsack method for stripe height selection. In all cases, both Christou and Martin [Chr96], [Mar98] did much better than the other methodologies. Christou [Chr96] also studied non-rectangular grids. Table 3 contains those results.

When looking at table 3, for Christou's methodology [Chr96] if the number of cells within the overall grid is held constant as the number of cells to be assigned to each component increases, performance decreases. Intuitively, this makes sense, since it is harder to assign a small number of bigger groups near-optimally than it is to assign a large number of smaller groups of cells near-optimally.

While the methodologies of Christou-Meyer and Martin [Chr96], [Mar98] outperformed the other methods, both methodologies have shortcomings (to be discussed later) that are eliminated by the results in this dissertation.

Problem		RSB		GEOMETRIC		MSP	
M x N	P	Time	Gap %	Time	Gap %	Time	Gap %
32 x 31	8	1.8	6.52	43.6	5.43	0.01	1.09
32 x 31	256	4.3	6.73	152.3	-2.73*	0.01	0
32 x 30	64	3.0	6.25	90.4	6.25	0.01	0
100 x 100	8	9.0	9.33	111.0	7.39	0.04	5.63
128 x 128	128	85.5	14.13	539.9	7.13	0.04	1.63
256 x 256	256	227.8	13.25	3304.2	4.15	0.09	0
512 x 512	512	-	-	-	-	0.25	0.14

Table 2: Martin versus other graph-partitioning algorithms for rectangular grids

1.4 Contributions of this Dissertation

Striping algorithms may be classified as two-phase processes. A set of stripe heights must be selected in the first phase. In the second phase the cells within the stripes are assigned (possibly with overflow into the next stripe at stripe ends). As a result of this research, both phases have been improved.

1.4.1 Local Optimality Assignments within Stripes

A partition that cannot be improved by swapping the assignments for exactly two vertices is said to be **locally optimal** (This is the smallest change that can be made in a feasible solution in order to obtain a different feasible solution, in the equi-partition case). By producing locally optimal solutions initially, the amount of post processing will be reduced, since there will be no need to check for improving two-cell swaps.

Problem		RSA		RSQ		DGA	
Shape	P	Time	Gap %	Time	Gap %	Time	Gap %
circle	16	23.3	24.44	9.1	21.80	19.8	8.33
circle	64	34.7	16.87	14.5	28.34	19.4	5.87
ellipsis	16	2.3	10.83	1.4	13.33	8.3	8.33
ellipsis	64	3.5	5.16	2.2	15.10	9.4	5.36
torus	16	27.3	28.97	12.5	32.67	18.8	11.50
torus	64	36.5	22.86	18.5	34.3	17.2	11.00
diamond	16	14.0	38.67	6.5	35.74	10.7	16.40
diamond	64	18.7	29.78	9.0	28.80	16.2	13.37

Table 3: Christou-Meyer versus other graph-partitioning algorithms for non-rectangular grids

By identifying conditions under which local optimality is achieved by greedy assignment procedures, a suite of alternative stripe-assigning procedures was developed to reduce undesirable assignments. While a given stripe-assigning procedure will not always produce a good assignment, for most instances, at least one of the procedures within the suite will work well.

1.4.2 Stripe-Height Selection

The bottleneck in all earlier striping algorithms was the stripe-height selection process. For these methods to obtain the best solution, at least a super-polynomial amount of time would potentially be required. An algorithm will be presented that produces an optimal set of stripe heights in polynomial time.

1.5 Organization of this Dissertation

In Chapter 2, an overview of the methodology of this dissertation will be presented, along with short descriptions of some well-known or related algorithms. Chapter 3 presents three algorithms that produce locally optimal solutions. Chapter 4 shows how a grid graph may be broken into a series of smaller and independent subproblems. Chapter 5 describes how to select the optimal set of stripe heights for a given assignment procedure. Chapter 6 contains results and comparisons with some previous methodology. Chapter 7 contains concluding remarks and directions for future work.

Chapter 2

Background and Related Work

2.1 Introduction

Unless the graph is small, standard branch-and-bound techniques aren't very useful for partitioning a graph. The amount of computing resources required grows far too quickly. Using AMPL [FGK93] to formulate the problem and CPLEX [Fra99]to solve it, the software can reasonably handle at most the case of a 5 x 5 grid to be partitioned among five components. For problems larger than this, the branch-and-bound trees get too large to handle. Space and time become limiting factors.

Several heuristics have been devised for discovering good, if not optimal, solutions. The algorithms to be discussed can be roughly broken into two groups: stripe-based and non-stripe-based. The algorithms resulting from the research presented in this dissertation, along with the related predecessor algorithms, fall into the category of stripe-based methods. The other algorithms fall into the class of non-stripe-based methods.

A general description of the stripe assignment procedures (defined in Chapter 3) used by Donaldson-Meyer (DM) and the formulation of stripe-height selection as a shortest-path problem or as a dynamic-programming problem will be presented. The latter two formulations suggest algorithms that exploit the presence of overlapping subproblems within the solution in order to eliminate redundant calculations. These two formulations are basically equivalent, except for the organization of data for the subproblems. The end result is that both find the best stripe-based solution in polynomial time (although, only the run-time for the dynamic-programming based solution will be presented).

2.2 Non-Striping Partitioning Algorithms

Four algorithms will be presented in this section: Kernighan-Lin, Recursive-spectral Bisection, Geometric Mesh Partitioning, and METIS. None of these algorithms take advantage of known geometric properties of a graph (Geometric Mesh Partitioning assumes that vertices that are far apart in euclidean distance probably don't have an edge connecting the two vertices.). This is the main difference between the non-striping and striping algorithms.

2.2.1 Kernighan-Lin

A well-known graph partitioning algorithm is due to Kernighan and Lin [KL70] [Chr96]. The algorithm divides the vertices into two groups and then looks for sequences of swaps of vertices that reduce the number of cut edges. Within this sequence of swaps, it is possible that a swap could be made that increases the number of cut edges, but leads to later swaps that produce a net improvement in the number of cut edges. This algorithm can lead to a locally optimal solution.

Kernighan and Lin proposed a way of getting away from a locally optimal solution. Assume that the original graph has been partitioned into two groups: A and B. Within both groups, recursively run the algorithm and identify groups A_1 and A_2, subgroups of A, and B_1 and B_2, subgroups of B. The algorithm can then be run again using some combination of subgroups as an initial partition.

2.2.2 Geometric Mesh Partition

The method of Miller, Teng, Thurston, and Vavasisi [GMT95], [MTTV93] differs from the other non-striping algorithms in that this algorithm doesn't use the adjacency information of a graph to partition the graph. Instead this algorithm partitions the vertices of a graph using only the physical location of vertices as the deciding factor. When describing the algorithm, the ideas will be presented using two- and three-dimensional terms, but the methodology is applicable for higher dimensions.

Geometric Mesh Partition Algorithm (Additional discussion about individual steps follows the description of the algorithm.)

Input - The (x,y) coordinates for every vertex in the graph.

Output - The vertices in the original graph are divided into three sets:A,B, and C, where A and B don't share an edge and C is a set that when removed disconnects all paths from A to B.

Algorithm

1. Map the points from R^2 onto a unit sphere in R^3.
2. Determine the centerpoint (defined below) for the points in R^3.
3. Rotate and scale the points so that the centerpoint is at the
   the origin.
4. Find a cutting plane that goes through the centerpoint.
5. Project down to R^2 the points and cutting plane from step 4
6. Determine sets A, B, and C

For step 1, for a given (x,y) pair in R^2, this point is mapped to (x,y,0) in R^3. The corresponding point on the unit sphere is calculated by projecting along the line determined by (x,y,0) and (0,0,1).

For step 2, the centerpoint is defined to be a point such that every plane through that point divides the points into approximately equally sized groups. The authors [GMT95] state that every set of points contains a centerpoint and the centerpoint can be found using linear programming.

For step 4, the cutting plane defines a “Great Circle” that lies on the the sphere and goes through the origin.

For step 5, the authors use several factors for partitioning the vertices into the three groups. In the actual inplementation, the vertices are divided into only groups A and B, thus creating an edge-separating set.

Although the algorithm doesn’t use any edge information, the authors [GMT95] apply the methodology in a way so as to glean some edge information. Instead of calculating the centerpoint, the authors calculate a “pseudo-centerpoint”. This is done by using a sample of points to calculate a “centerpoint”. For this calculated centerpoint, several cutting planes are generated. The cutting plane that performs the best is the one that is used.

2.2.3 Recursive Spectral Bisection

Recursive spectral bisection describes a class of algorithms that follow the general form [ST93]:

Function Recursive Bisection

Input - A graph G = (V,E) and p = number of groups (p = 2^n)

```
1. Find an "optimal" bisection, $G\prime$ and $G''$ of G.
2. While (| $G\prime$ | > |V|/p)
        Perform Recursive Bisection($G\prime$)
        Perform Recursive Bisection($G''$)
3. Return the subgraphs $G_1$,$G_2$, ... , $G_p$
```

Finding an "Optimal" Partition: Spectral Bisection

Assume that the graph G = (V,E) is to be partitioned. For every vertex v_i in V create a new variable x_i that may assume the values 1, -1. If x_i is not equal to x_j, then v_i is not in the same group as v_j. The problem of finding the minumum number of cut edges for partitioning a graph into two parts can be modeled as the following quadratic program [HL95b], [Chr96].

$$\min \tfrac{1}{4}\textstyle\sum_{(i,j)\in E} (x_i - x_j)^2$$

$$s.t. \begin{cases} \sum_{i=1}^{|V|} x_i = 0 & & (1) \\ x_i \in \{-1, 1\} & i = 1..|V| & (2) \end{cases}$$

Define L, the *Laplacian* matrix of the graph, to be:

$$L_{i,j} = \begin{cases} -1 & if \;\; (i,j) \in E \\ d_i & if \;\; i = j \\ 0 & otherwise, \end{cases}$$

where d_i is the degree of of vertex v_i. The matrix L = D - A, where D is a matrix such that D_{ii} equals the degree of v_i and all other entries are zero, and A is the adjacency matrix of G. The matrices D, A and the previous objective function are related as follows:

$$\begin{aligned}\textstyle\sum_{(i,j)\in E} (x_i - x_j)^2 &= \textstyle\sum_{(i,j)\in E}(x_i^2 + x_j^2) - 2 \sum_{(i,j)\in E} x_i x_j \\ &= \textstyle\sum_{(i,j)\in E} 2 - x'\mathrm{A}x \\ &= x'\mathrm{D}x - x'\mathrm{A}x \\ &= x'\mathrm{L}x.\end{aligned}$$

The original quadratic problem then becomes:

$$\min \tfrac{1}{4}\, x'\mathrm{Lx}$$

$$s.t. \begin{cases} x'e & = 0 & (1) \\ x_i \in \{-1, 1\} & i = 1..|V| & (2) \end{cases}$$

This problem is NP-Complete.

In the previous problem, constraint (2) can be relaxed producing a new problem that has a known solution. In the previous formulation, if constraint (2) is replaced by the constraint x′x = n, the optimal solution for the relaxed problem is x = $\sqrt{n}$(second normalized eigenvector of L) (this result was stated in [HL95b]). This produces $\mathbf{x} \in R^n$, but the components of **x** may not be feasible for the original quadratic problem.

To generate a feasible point, first, the median of the components of **x** is calculated. For each component of **x**, if x_i is greater than the median, then the corresponding vertex is assigned to group 1; otherwise, the vertex is assigned to group 2. Some evening out of the groups may be required, to obtain partitions of equal size.

2.2.4 METIS

METIS is the name of the software implementation of the methodology of Karypis and Kumar [KK95c], [KK95a], [KK95d]. A high-level view of the methodology is:

1. Collapse the original graph down to a smaller graph through a series of matchings. Matched vertices are combined as a single vertex as are the corresponding set of edges that were incident to the vertices.
2. Partition the reduced graph (for example, by spectral bisection).
3. Expand out the smaller graph, while maintaining the general partition created in step 2.

After step 1, one vertex may represent several vertices. If vertex v is is one of the vertices in the final collapsed graph and is assigned to group 1, then all of the vertices that v represents are initially assigned to group 1. The speed of this algorithm comes from the fact that size of the graph that is actually partitioned may be substanially smaller than the original graph. At each phase of expansion, reassignment of vertices may take place.

In the actual implementation of METIS, the authors implement four different algorithms for partitioning a graph. Three of the algorithms are based on graph growing heuristics. The other algorithm is based on spectral bisection. The authors evaluate the performance of the different algorithms in [KK95b].

2.3 Stripe-Based algorithms

For a given grid, a stripe-based algorithm partitions the rows into groups of consecutive rows (stripes) and then **stripe-wise assigns** the cells, within the stripes, to components. Figure 4 presents a graphical description of this process in which cells are consecutively assigned (top-to-bottom in each column) until a component has the correct number of cells.

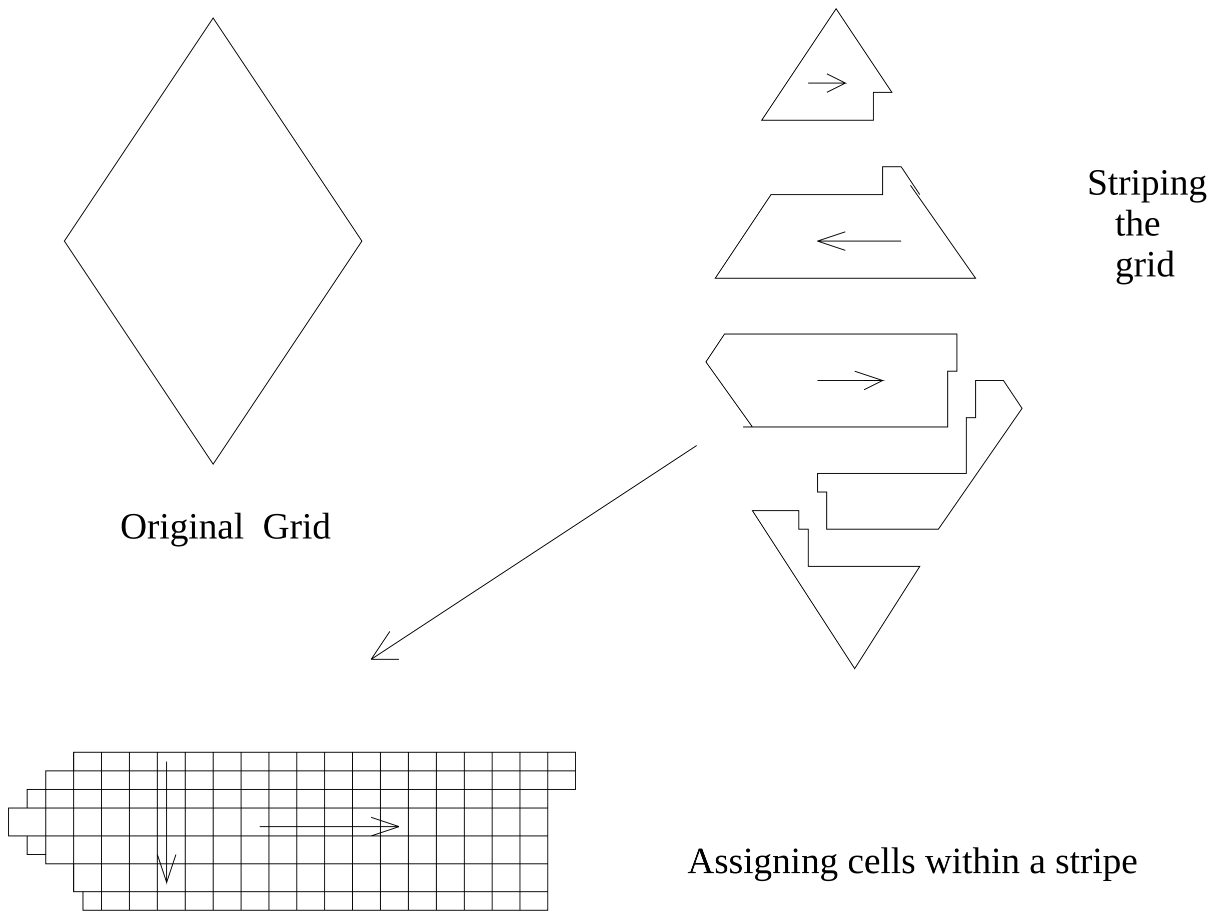


Figure 4: Stripe-based assignment of a grid.

An **optimal component** is defined to be a component whose perimeter equals the minimum perimeter required to enclosed the corresponding number of cells (area). Yackel and Meyer showed that for every given number of cells a rectangle, with possibly at most two fringe columns (columns that contain fewer assigned cells than all the other columns, except perhaps for another fringe column), is an optimal component. A stripe height is said to be **optimal** if optimal components can be assigned within the stripes.

For optimal or near-optimal stripe heights, stripe-based fill procedures produce optimal or near-optimal components (in most cases, more on this later). This follows from the fact that the stripes provide an easy way to assign rectangularly-shaped components. This discovery of organizing the assignments into stripes came about from the use of genetic algorithms for generating feasible solutions.

2.3.1 Stripe Assignments based on Genetic Algorithms

Both the methodology of Yackel-Meyer and of Christou-Meyer use a genetic-algorithm heuristics [Gol89] [Hol92] [Mic94] for generating feasible solutions. Genetic algorithms create a new set of feasible solutions from a parent set. The interactions of combining parts of two feasible solutions (crossover) or changing part of a feasible solution (mutation) are used to generate new feasible solutions. Both Yackel-Meyer and Christou-Meyer then add in a decision-making policy for throwing out certain feasible solutions (survival policies).

The major difference in the methodologies of Yackel-Meyer and Christou-Meyer is how the elements of the feasible set of solutions are discovered. For Yackel-Meyer, there is a single processor (host) that maintains the current set of feasible solutions. The host node distributes work to the "node" processors. For Christou-Meyer, each node works on its own set of feasible solutions. The nodes interact when the processor broadcasts a set of feasible solutions to the other processors and then receives sets of solutions from the other processors.

2.3.2 Yackel-Meyer Algorithm

Yackel and Meyer implemented a parallel version of a genetic algorithm. In this model, there are two types of processors: host and node. The host processor coordinates activities by sending out pairs of feasible solutions to node processors. A node processes the feasible solutions via crossover to produce a pair of offspring. The offspring are then mutated. A pair of solutions (selected from the two parents and two offspring) is then passed back to the host processor. The process continues until stopped by a limit on the number of iterations or if the lower bound is hit. The user is not guaranteed an optimal solution, unless a feasible solution that matches the lower bound is found.

2.3.3 Christou-Meyer Algorithm

Christou and Meyer used several different ways for generating feasible solutions (a randomized method for generating stripe heights and methods based on genetic algorithms). The results generated using a distributed genetic algorithm produced the best overall results and will be discussed here.

Christou and Meyer eliminate the need for a host processor. From a very high-level point of view, Christou-Meyer starts up a certain number of sibling processes each of which generates a set of feasible solutions. At each node, certain members of the feasible set are mated (and on certain individuals other genetic operations are applied). Periodically, each process selects individuals to share with other processes and receives individuals from other processes. This loop continues for some specified amount of time.

Christou and Meyer showed that any grid with enough rows can be partitioned into stripes that allow the assignment of components that are of optimal or near-optimal perimeter, as defined by Yackel and Meyer (see lemma 3 in [Chr96]). It follows that for a large group of grid graphs, the vast majority of components will have perimeters at or near optimal value. Figure 5 illustrates this point.

In figure 5, those components labeled I (I is for interior-to-stripe) are at or near optimal perimeter. The unmarked components may be far from optimal. If the growth in the perimeters for the unmarked components can be controlled, the total perimeter for the grid will be very good. Intuitively, it seems reasonable that this striping methodology should produce very good results. Christou and Meyer have proven, under mild assumption, that stripe-based solutions asymptotically approach (from the standpoint of relative error) the optimal perimeter.

In particular, Christou and Meyer showed under mild assumptions (for MxN rectangular grids partitioned among P components) that the relative difference between the observed performance and the optimal partition (defined as δ below) is bounded from above as follows:

$$\delta < \frac{1}{A_p^{0.5}} + \frac{1}{A_p}.$$

As A_p, the number of cells assigned to component p, tends to infinity(as also does the number of cells in the grid), then the relative difference goes to zero. For more details, see Chapter 3 (Theorem 8) in this dissertation or Chapter 2 in [Chr96].

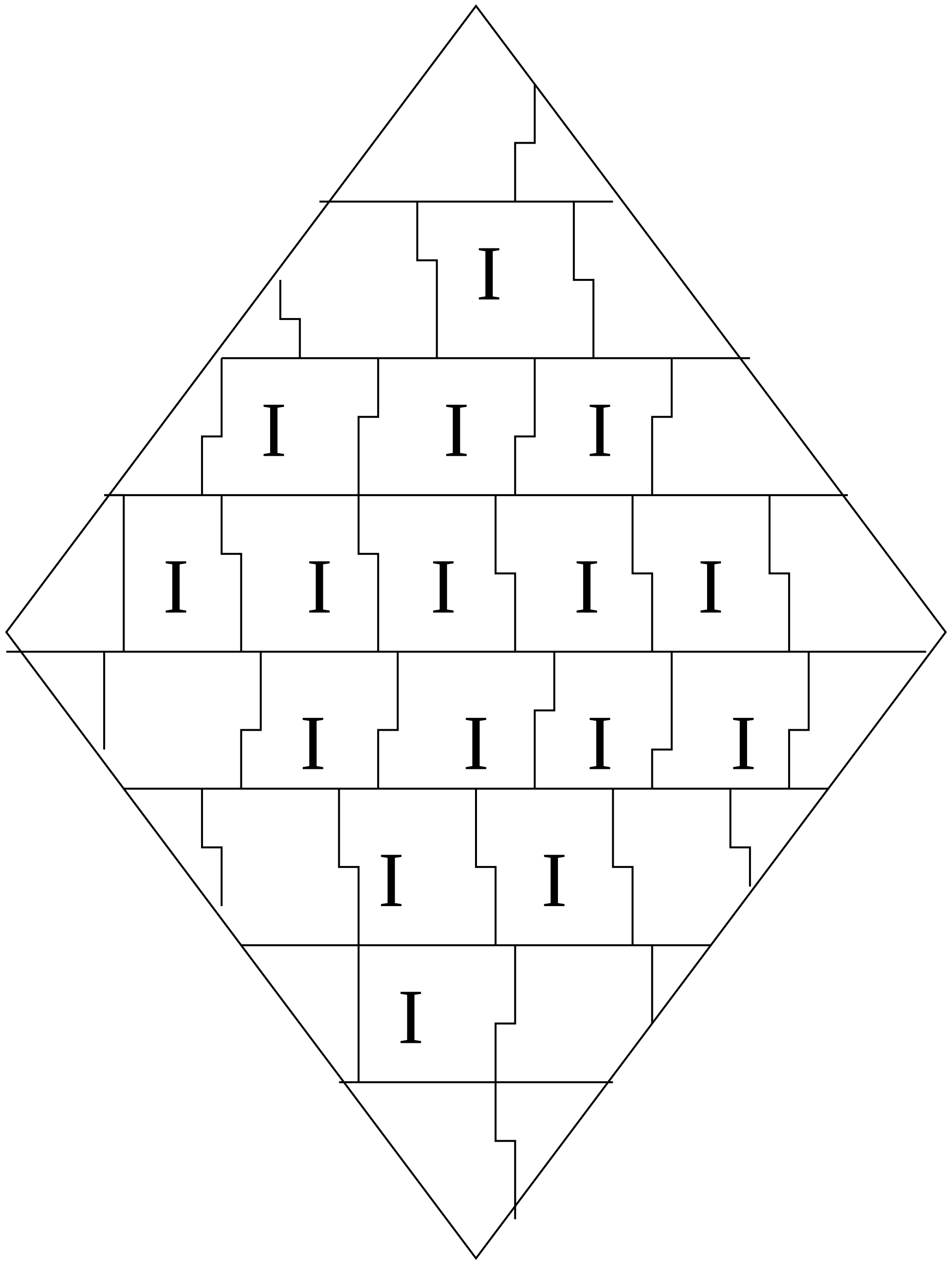


Figure 5: Interior-to-stripe (I) components at or near optimal perimeter

2.3.4 Knapsack Algorithm

Garey and Johnson in [GJ79] discuss the Knapsack problem and show it is an NP-Complete problem. The following definition was taken from page 134 of [GJ79]:

Given : Given a finite set of items, **U**, such that each u $\in$ **U** has an associated size s(u) $\in Z^+$ and value v(u) $\in Z^+$ and a positive knapsack capacity B $\geq$ max {s(u): u $\in$ U}.

Solve

$$\max \sum_{u \in U} v(u)$$

$$\sum_{u \in U} s(u) \leq B$$

Martin [Mar98] converts the stripe partitioning problem into a slight variant of the Knapsack problem. In this variant formulation, the sum of the weights must equal B (Martin also solves the minimization version of this problem.). This alternate problem is also NP-Hard, since the original Knapsack problem can be reduced to this modified Knapsack Problem by adding a slack variable.

Martin used software [MT90] that required that the problem be in standard form (i.e. the constraint being an inequality). For a restricted set of inputs, Martin showed that the solution of the problem in standard form satisfied the inequality constraint as an equality. Because of this restriction on the inputs, we do not know if the problem that Martin solved is NP-Hard.

For Martin's algorithm only rectangular grids are considered. Only stripe heights corresponding to an integral number of components are allowed. (That is, components are not allowed to overflow from one stripe into the next.) Because of these assumptions, the author is able to take advantage of certain features of the original graph. For a more detailed discussion of the material that follows see Chapter 5.

For each valid stripe height, there is a unique perimeter (the objective coefficients in the previous narrative) determined by adding the perimeters of the components associated with the height (the s(u)'s in the previous narrative). Since the domain is a rectangle, for a given stripe height, no matter where these rows appear within the grid, the same number of cells will occur in the stripe. It follows that for a feasible solution, the order of the stripe heights doesn't matter and the number of feasible solutions involves counting combinations, rather than permutations. However, the total number of solutions is at least super polynomial, as will be proved in Chapter 5.

2.3.5 The Shortest-Path/Dynamic-Programming Approach

Both methods are based on the fact that, under certain assumptions, two row indices of the grid and a direction of assignment can be used to divide the original problem into two problems that are identical in nature to the original problem, but which can be solved independently of one another. (These two smaller problems will be referred to as **subproblems** throughout the remainder of this dissertation) The ability to divide the original problem into similar, but independent problems, is the single most important tool in the discovery of polynomial-time algorithms for determining the best stripe-based solution.

Although there are two algorithms presented, the only real difference between the methods is how the the data for subproblems are organized.

2.4 Improved Stripe Assignment

Donaldson and Meyer [Don97] show that under certain circumstances a modified version of the Christou-Meyer fill produces a locally optimal solution for a rectangular grid. Donaldson and Meyer also present two algorithms that can produce locally-optimal solutions for a larger class of grids.

Initial experimentation using a slight variation of one of the locally-optimal algorithms (the Improved U-turn Algorithm, see Chapter 3) did not produce very good results. As a result, research into different assigning procedures was conducted that produced a set of nine different assignment methods for the cells that are assigned to components appearing in more than one stripe. This multi-heuristic approach produced very good results.

2.5 Summary

When looking at a stripe-based method two areas for concern appear. Assignments of components that cross stripe boundaries can add significantly to the total perimeter of the domain. Poorly chosen stripe-heights can also drastically affect the total perimeter. Methods that address these two issues are the two main areas of focus for the remainder of this dissertation.

Throughout the remainder of this dissertation, references will be made to the work of Yackel-Meyer and Christou-Meyer. Rather than write out the full names, when the work of Yackel and Meyer is being referred to, **YM** will be used. In the case of Christou and Meyer, **CM** will be used.

Chapter 3

Improved Stripe Assignment

3.1 Introduction

In earlier research, Yackel-Meyer (YM) and Christou-Meyer (CM) were able to produce very good results using striping techniques. Nothing was known about the possibilities for improvement via swapping (although, in their implementation, CM used a post-processing swap phase). In particular, it was not known under what conditions when pairs of cells could have their assignments swapped with the result being an improved total perimeter.

The first result of this research shows that under certain conditions (including rectangular origin domain) CM does produce a locally optimal solution (i.e., an assignment that can't be improved by reassigning two cells). Local optimality is clearly desirable from a theoretical viewpoint. It is also computationally desirable since it eliminates the need for a post-processing swap phase. It will also be shown that CM can also make assignments that are far from locally optimal, if certain conditions are not satisfied.

We now discuss the **Basic U-turn algorithm**, and show that it makes assignments that are guaranteed to be locally optimal for all cases of rectangular grids. This method can be extended to a more elaborate algorithm, the Improved U-turn algorithm. The **Improved U-turn algorithm** will be briefly discussed but the proof of its local optimality will only be sketched.

The concept of a U-turn region (to be defined later) arose initially as we considered the area within an assignment where swap improvements could be made. As mentioned in the previous chapter, YM and CM make assignments such that the majority of components have optimal or near-optimal perimeters. However, large deviations from optimality occurred when trying to assign components at the boundary of the domain. The research presented in this chapter is designed to reduce the effects of these boundary components.

Two factors were examined to reduce the ill effects of a poor assignment within the U-turn region. The first was an improvement of the CM method called the Basic U-turn algorithm. The second factor involved increasing the size of the U-turn region and developing assignment patterns that would be better suited to handle certain situations (the Improved U-turn algorithm). The reader should notice that by expanding the U-turn region, components that would have been assigned in near-optimal patterns may no longer be, so a balance between the size of the U-turn region and the number of well-shaped components had to be achieved.

Local optimality of a fill procedure is proved only in the case of rectangular domains. However, the ideas developed in conjunction with the proof are useful in terms of developing good fill procedures for more general domains as we will see in Chapter 6.

The U-turn region also provided a completely unexpected result. This region in the grid provided a convenient method of dividing the grid into independent parts. This ability to divide the grid into independent parts is the foundation upon which the second major breakthrough of the research is built. This led to the discovery of a polynomial algorithm that produces the best stripe-based solution for a given fill procedure.

3.2 Terms and Definitions

In this chapter, only **rectangular grids** will be considered. Examples of a **rectangular** and a **non-rectangular grid** are given in figure 6.

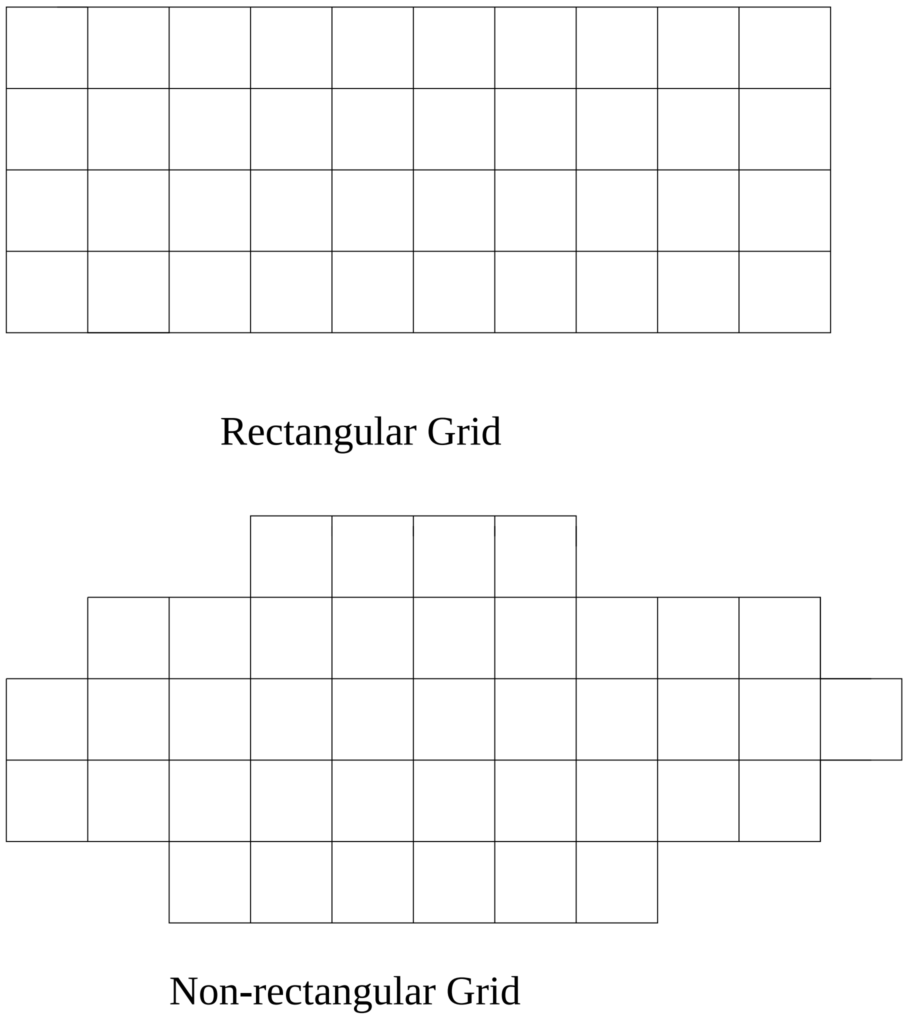


Figure 6: A rectangular and a non-rectangular grid

Definition - A **stripe** is defined to be any collection of consecutive rows within a grid.

Definition - A **component** is any collection of cells assigned to the same group (or processor). Figure 7 shows a component assigned to group A.

Definition - The process of assigning the cells within a stripe is called **stripe assignment**.

Definition - For this discussion, a solution is said to be **locally optimal** if the overall perimeter can't be reduced by swapping the assignments for two cells.

Definition - Swapping the assignments for two cells will be referred to as a **two-cell swap**. (We focus on the balanced case in which components have equal area. In this case the smallest change that can produce another feasible solution is a two-cell swap).

Figure 7 will be used to demonstrate several definitions.

In figure 7, we assume that the cells shown represent all the cells assigned to component A and categorize the cells according to their contribution to the total perimeter of the component.

Definition - An **interior cell** is assigned to the same component as all four of its neighbors. In figure 7, interior cells are marked as 0.

Definition - An **edge cell** contributes one to the perimeter. These cells are marked with a 1 in figure 7.

Definition - A **vertex cell** is a corner cell in a component or a cell with exactly two neighbors assigned to its component. These cells are marked with 2's. Type 2 cells may also occur in "peninsulas". These are marked with 2^* in figure 7.

Definition - A **spike cell** is a cell that has only a single neighbor assigned to the same processor. This cell is marked with a 3.

Definition - An **island cell** is assigned to a different processor than all of its neighbors. An example of this is marked with a 4. (In the constructions to follow, island cells are not generated)

Definition - In figure 7 those cells marked as 2^* and 3^* make up a **peninsula**. A peninsula is a connected collection of cells with the property that all cells in it are type-2

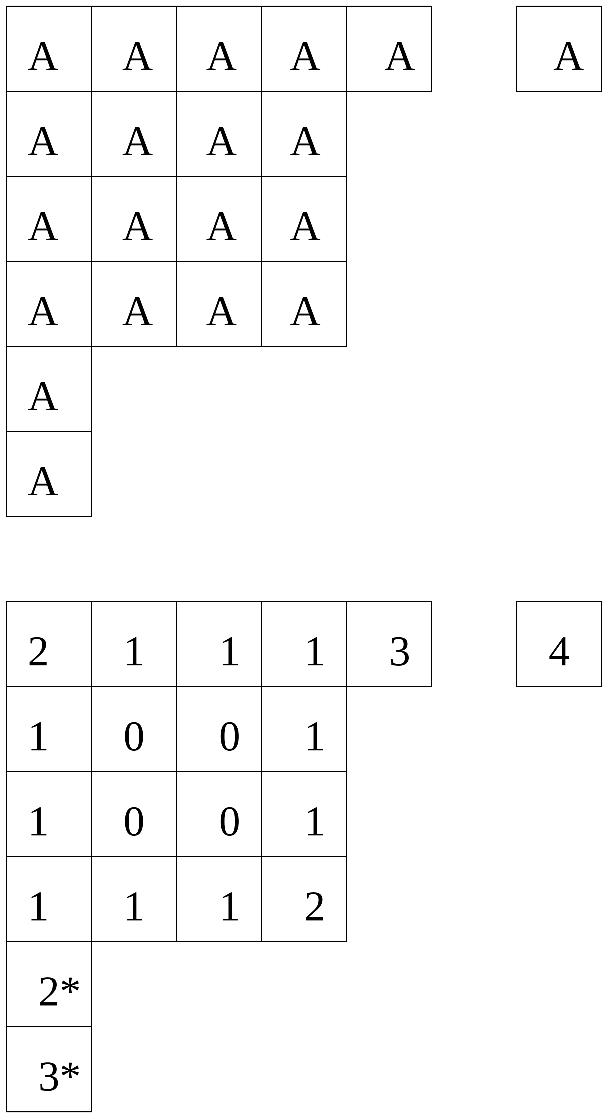


Figure 7: Classification of cell assignments by perimeter contribution

or 3.

Definition - **Boundary cells** are those cells that are of types 1 through 3

Definition - **Semi-perimeter** equals the sum of the number of rows and columns occupied by a component.

Definition - The **enclosing frame** (also known as the **rectangular hull** or **enclosing rectangle**) for a connected component is the minimum sized rectangle that encloses the component.

Definition - If a component is not a rectangle, but can be represented as the union of a rectangle plus additional incomplete "boundary" rows or columns, then the cells in these "boundary" rows or columns are **fringe cells**.

Definition - A component is said to be **slice convex** if for any two cells within a row or column of a component, all the cells within the row or column between these two cells are assigned to the component.

Definition - For a component C, a **gap** occurs within a column (or row) if there are two cells, *a* and *b*, assigned to C within that column (or row), and one or more cells between *a* and *b* that are not assigned to C (between *a* and *b* there are no other cells assigned to C). Figure 10 shows examples of interior and boundary gaps.

Definition - A component is said to be top-to-bottom **column-wise** assigned if, with the rows numbered top to bottom and the columns numbered left to right, within column j, cell[i][j] is assigned before cell[i+1][j] and cells in column j will be assigned before cells in column j-1 (when assigning right to left) or j+1 (when assigning left to right).

Definition - To **row-wise** assign a component is to assign all cells in row i, either left to right or right to left, within certain columns, before assigning any cells in row i+1.

3.3 Local Optimality of the CM Fill Procedure

In figure 8, we have two locally optimal solutions. The assignment on the right reflects the simplest columnwise fill procedure for a single stripe that we will consider in part 2. For any component, the perimeter cannot be reduced by a two-cell swap, without making another processor's perimeter worse by a larger or equal amount.

1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4

1	1	2	3	4	4
1	1	2	3	4	4
1	2	2	3	3	4
1	2	2	3	3	4

Figure 8: Locally optimal assignments

Lemma 3.1 *The perimeter for a connected component is greater than or equal to the perimeter of the enclosing frame. If the component is slice convex and connected, then the two perimeters are equal. Perimeters can be calculated using the following formulas; where the second formula applies even if the component is not connected.*

(1) **Perimeter** **= 2 * (no. of rows + no. of cols**
(slice-convex component) **= 2 * semi-perimeter**

(2) **Perimeter** **= 2 * (no. of rows + no. of cols**
(non-slice-convex component) **+ no. of gaps)**
= 2*(semi-perimeter+no. of gaps)

Proof

If the component is slice convex, then there is a 1-1 and onto mapping between the perimeter edges of the component and the edges of the enclosing frame. Figure 9 contains an example of a slice convex and non-slice convex domain. If the enclosing frame for the domain is broken up into unit lengths, where each length is considered an element, then there is an obvious 1-1 and onto mapping from this set of elements and the perimeter edges of the slice-convex domain. The mapping would project each edge of the grid to the corresponding edge of the enclosing rectangle (see edges marked e,f,g, and h in figure 9).

For the case of a non-slice convex component, the mapping is no longer an isomorphism, because the number of perimeter edges for the domain is greater than the number of elements in the set of unit lengths for the enclosing rectangle; but the mapping from the edges of the enclosing frame to the perimeter edges of the component is 1-1.

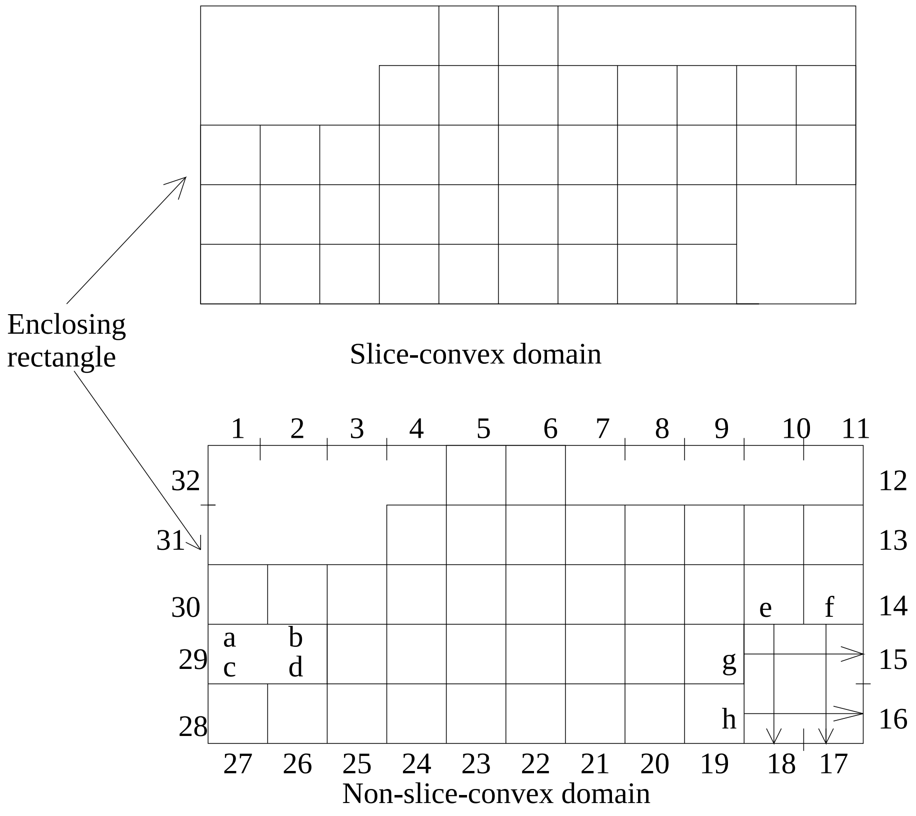


Figure 9: No elements in the enclosing frame map to a,b,c, and d

A non-slice convex component has gaps within rows or columns which contribute to the total perimeter. If the gap is along the boundary of the grid (that is, the gap occurs in a row or column but not both), then two additional edges are added to the set of perimeter edges of the component. If the gap is in the interior of the grid, then the gap occurs in both a row an column and four additional edges are added to the set of perimeter edges (see figure 10) (The gap count in Lemma 3.1 includes both row and column gaps.).

□

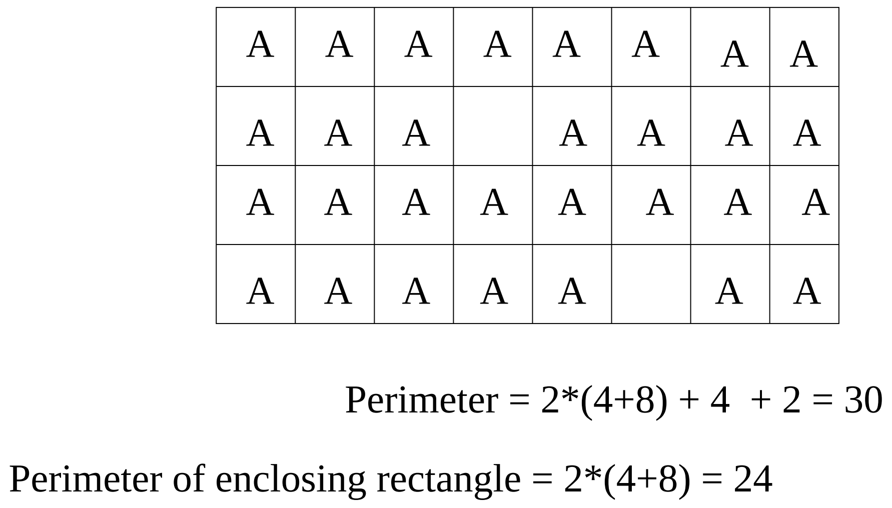

Perimeter = 2*(4+8) + 4 + 2 = 30
Perimeter of enclosing rectangle = 2*(4+8) = 24

Figure 10: Perimeter for component with interior gap and boundary gap

Lemma 3.2 *Adding a cell to a slice-convex component cannot decrease the perimeter.*

Proof

Follows from Lemma 3.1, formulas 1 and 2, since semi-perimeter cannot decrease.

□

Lemma 3.3 *The perimeter for a rectangular component can not be reduce by a two-cell swap.*

Proof (sketch) - Case 1, the component is assigned to either a single column or row (see any of the components in the lefthand picture in figure 8). Moving a corner cell will reduce the number of rows or columns by one, but will increase the number of columns or rows by one. There is no improvement. Moving an interior cell increases the overall perimeter.

Case 2 - The component appears in multiple rows and columns. Moving a cell will not reduce the number of rows or columns, because each row and column contained more than one cell. In fact, wherever the new assignment is made at least one column or row will be added to the size of the enclosing frame.

□

In figure 11, we have a non-locally-optimal solution. Here the boldfaced 3 and 4 can be swapped and the overall perimeter will be reduced. In this case, moving the 3 reduces the number of columns by one. Moving the 4 to 3's old position doesn't make worse 4's perimeter.

Lemma 3.4 *Assuming slice convexity and no island cells, the only swap that can improve the total perimeter is one in which a spike cell is moved to a corner destination (one with vertical and horizontal neighbors in the same component).*

Proof -

Moving a type-3 cell to a corner position will reduce semi-perimeter because its origin was the only cell in its row or column and its destination is a position for which the corresponding row and column are already included in the semi-perimeter.

Type-k cells, k = 2,1, or 0, either

1) have vertical and horizontal neighbors in their component, so corresponding location swaps cannot reduce semi-perimeter; or

2) lie on a peninsula, in which case a swap produces a gap, and therefore, by the second formula in lemma 3.1, this gap compensates for the row or column count decrement and the perimeter cannot decrease.

□

1	1	1	2	2	2
1	1	1	2	2	2
1	1	1	2	2	2
1	1	**3**	3	2	2
4	4	4	3	3	3
4	4	4	3	3	3
4	4	4	**4**	3	3

Figure 11: A non-locally optimal assignment

In the original CM fill method, the actual columns were filled from the top. We will show that a modified CM fill method will produce a locally optimal solution if certain conditions are met, as indicated in the following theorem:

Theorem 3.1 *Let the following assumptions be satisfied:*

1. *Graph partitioned using a stripe decomposition method.*
2. *Filling by column, alternate (down and up) fill directions.*
   (snake-fill procedure).
3. *An integral number of processors is contained within each stripe.*
4. *Component area is ≥ 4.*
5. $\frac{3}{4}$ ** component area $\geq$ the largest stripe height*

Then the solution generated is locally optimal.

(In the discussion to follow, **A** will be used to denote the area or the number of cells assigned to a component.)

The first three conditions are needed for defining the stripe assignment. The last two are needed for technical reasons. If the area to be assigned to a component is 1, 2 or 3, then any connected component is of minimum perimeter, so the assignments produced by the fill procedure are actually optimal. So we only consider areas greater than or equal to 4. As a consequence of assumption 5 we will show that there will never be spike cells in consecutive columns.

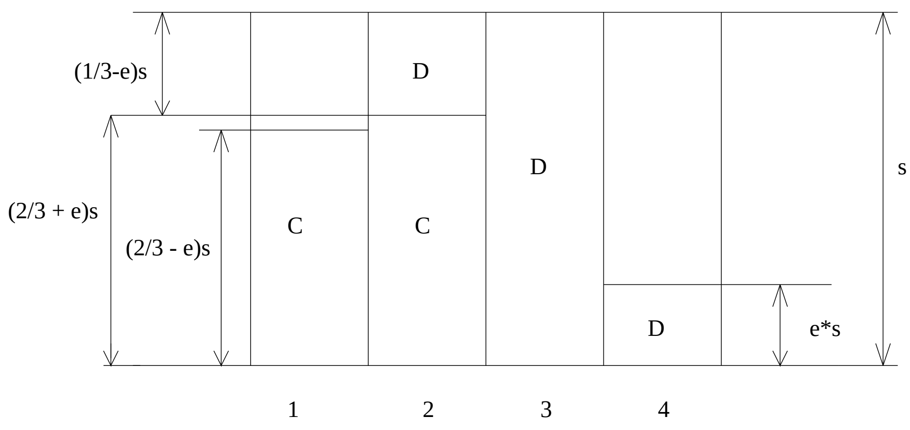


Figure 12: Spike cells in non-consecutive columns

In figure 12 we have two components, C and D, with area equal to 4/3 of the stripe height (The proof is analogous if A > 4/3*stripe height). If C is to have a spike in the column labelled 2, then the number of C cells in column 2 must be greater than the number of C cells in column 1. Thus, we assume that C contains (2/3 - ϵ)*s cells in column 1 and (2/3 + ϵ)*s cells in column 2. The remainder of column 2 contains (1/3 - ϵ)*s cells of D. Since D is also assigned an area equal to 4/3*s, there are s+ ϵ remaining cells of D to be assigned. That means that all of the column labelled 3 and part of 4 will contain D's. Therefore, D cannot have a spike cell in column 3, and D and C cannot have spike cells in consecutive columns (This prevents the following from happening: suppose s > 3/4*A and as a result, there are no D cells in column 4, then the bottom D cell in column 3 could be swapped with the top C cell in column 2, and the total perimeter would be reduced.)

Observations

Fact 3.1 *Assumption 2 guarantees that the cells for a given processor are connected. This follows from the fact that the first cell assigned in the next column will be adjacent to the last assigned cell in the current column.*

Fact 3.2 *The assignment is slice convex. This follows from the fill procedure. Within column, the cells are assigned consecutively. Columns are filled from left to right, and this implies that the row assignments are slice convex, since clearly this fill can produce no gaps in a row.*

Fact 3.3 *It follows from Fact 3.1 and assumption 5 that no component will contain an island cell.*

Overview of Proof of Theorem 3.1 - The proof will be broken into several parts. First, certain cells will be eliminated as possible candidates for swapping. Across-stripe swaps will be shown to not improve the overall perimeter. Lastly, it will be shown that no within-stripe swap will improve the overall perimeter.

In all cases, we will show that either:

1. The component has at least two cells within every row or column, which means swapping will not reduce the number of rows or columns of the component; or
2. The component has a single cell appearing in a row or column, in which case swapping that cell assignment would increase the perimeter of the other component.

	B	B*	A*	
...	B	B*	A*	...
	B*	A*	A	
		.		
		.		
		.		
	B*	A*	A	

Figure 13: Boundary cells: B^*'s and A^*'s

The following lemma is useful.

Lemma 3.5 *No swaps between non-adjacent components can improve the overall perimeter.*

Proof - Follows from lemma 3.4..

□

This lemma implies that the only cells that could possibly produce a better solution are those cells along the boundary between two adjacent areas.

Lemma 3.6 *A swap across a stripe cannot improve the combined perimeter for the components involved.*

Proof -

Such a swap cannot have a corner cell as a destination, by lemma 3.4 there can be no improving swap.

□

This lemma implies that only swaps of cells contained within the same stripe need be considered.

It will now be shown that there does not exist a within-stripe swap that improves the overall perimeter. If a swap were to improve the overall perimeter, for at least one of the components the perimeter would have to decrease and for the other component the perimeter would have to remain the same or decrease. (This follows from the fact for a given component that the only increments for change in size of perimeter are a decrease by 2, no change, and an increase by 2 or 4). Denote a component for which the size of the perimeter decreases as "I" (for improving). A component for which the size of the perimeter is at worst unchanged will be denoted by "N" (non-increasing).

In order to prove that there does not exist a within-stripe swap that reduces the overall perimeter, it will be shown that there does not exist a pair of adjacent components one of which is a N and the other is an I (called an I-N pair). Only adjacent regions need by considered by lemma 3.5

Because of the hypothesis that stripe height is less than or equal to 0.75*component area, a component cannot fit into a single column. Therefore, there are two cases that must be examined:

1. I intersects at least three columns.
2. I intersects exactly two columns.

Case 1 - A three-column I component

	I	I	N	
	?	I	N	
		.	N	
		.	N	
		.	N	
	?	I	I	
	1	2	3	column labels

(Note: The ?'s indicate that the cell may or may not contain an I.)

Also assume that an N component appears in column labelled 3. Since there can be no spikes in column labelled 2 because every cell in the column has at least two neighbors, the only swaps that would reduce the size of the perimeter of I are those that move a spike cell from column 1 or 3 to column 3 or 1. Assuming that a spike appears in column 3, then the swap to reduce I would not involve the N component. Therefore, this pair of components can not be an I-N pair.

For the case that I intersects more than three columns the argument is analogous (the only change is the number of columns completely assigned to I's).

Case 2 - I is a two-column component. (this is relevant when $\frac{4}{3}$*stripe height $\leq$ A $\leq$ 2 * stripe height).

We have the following case:

		I	I	
		I	I	
		H	I	
		.		
		.	?	
			I	
	1	2	3	column labels

Figure 14: I as a two-column component.

If A < 2*stripe height, by Lemma 3.4, if I's perimeter is to be reduced, then the spike cell must be moved to a corner position. In figure 14, that corner position is marked with an H. Whatever component was originally assigned in position H does not appear in column labelled 3. As a consequence of assumption 4, we know that cell H is not a spike cell. So to swap I to position H will reduce I's perimeter, but will also increase the other component's perimeter by an equal amount. Therefore, no overall improvement occurs.

Again, in the previous argument, if the fringe is on the other side, the argument still holds.

If A equals 2*stripe height, then all the components are rectangles and are at a local minimum.

All possible configurations that the I component could have assumed have now been checked, and no I-N improving swap is possible.

□

The left grid in figure 8 illustrates a local minimum of poor quality. In that example, each component has a perimeter of 10, whereas an optimal solution uses 2x2 components with perimeter equal to eight each.

However, if the CM method is applied with properly chosen stripe heights, good solutions are obtained. If the grid is MxN and the grid is to be broken into P partitions, then we have the following theorem based on constructing a feasible solution via a striping approach of CM [CM95].

Theorem 8 (Christou-Meyer) - Assuming P divides MN and that P $\geq$ max (M,N) the minimum perimeter problem MP (M,N,P) has a feasible solution whose relative distance δ from the lower bound satisfies:

$$\delta < \frac{1}{A_p^{0.5}} + \frac{1}{A_p}$$

Thus the error bound δ converges to zero as A_p (the area of each processor) tends to infinity.

3.4 Overflow Assignments

All the previous results are for rectangular grids that have been partitioned into stripes that can be assigned to an integral number of components. How does the CM algorithm handle the case that an integral number of components can't be assigned within a stripe?

When a component overflows from one stripe to the next, CM row-wise assigns the overflow cells. This can lead to the creation of peninsulas. Figure 15 shows two peninsula examples.

Component G has a horizontal peninsula, and component F has a vertical peninsula. Obviously, row-assigning the overflow cells does not always produce good assignments (to improve the assignments the reader can think of “folding in” the peninsula like a blade in a pocket knife). Column-assigning the cells can also produce peninsulas. We may now formally define the region in which these overflow assignments occur.

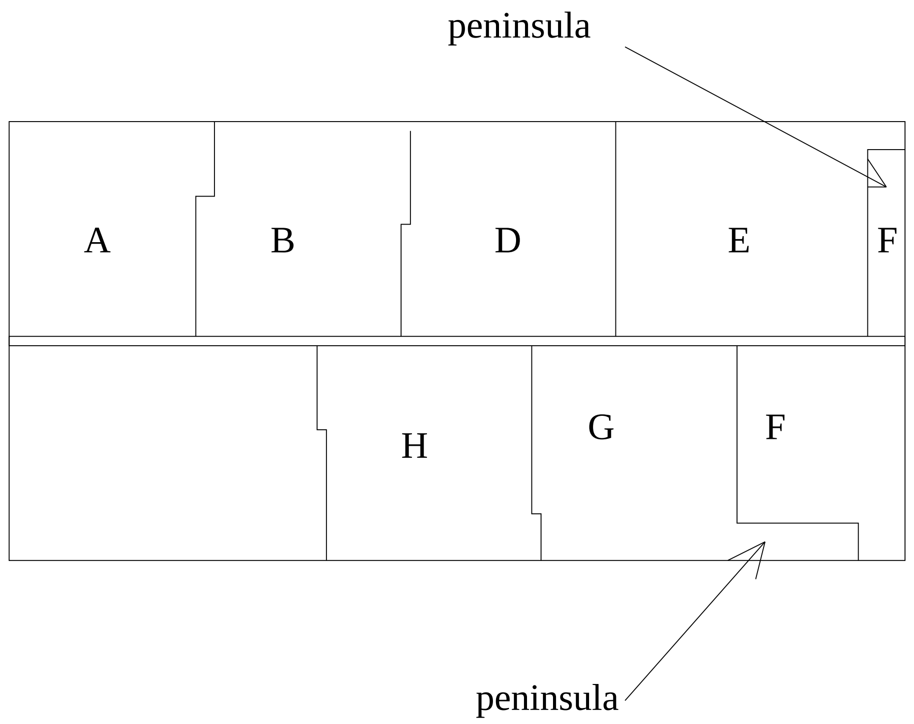


Figure 15: A horizontal and a vertical peninsula

Definition - Those cells at the end of stripe i that are assigned to components appearing in two stripes make up the **U-turn region** for stripe i.

In figure 15, those cells in the upper stripe assigned to F make up the U-turn region in this example.

In the next section the Basic U-turn algorithm will be presented. This eliminates certain peninsulas via a series of two-cell swaps and and reduces in the overall perimeter. The result is a local optimum.

3.5 The Basic U-turn Algorithm

Assumptions/Notation

For a given stripe i, an integral number of processors cannot be assigned to cells within that stripe. The last processor that is completely assigned within this stripe is N.

In the following algorithm, the direction of assignment is left to right; the arguments can be suitably modified if the direction of assignment is right to left. All columns are filled top-down (This is not a source of difficulty with respect to local optimum because we assume stripe height $<$ A/4.).

Overview of Algorithm

This algorithm can be broken into two parts. The first part is the initial columnwise assignment of cells in the grid. The second part searchs the grid for pairwise swaps that will reduce the overall perimeter. After all such swaps are identified and made, the result is a locally optimal solution. In figure 16 assignments are made columnwise top to bottom. The arrows indicate the direction of fill. Figures 17 and 18 show the kinds of swaps that are made as needed (the reader may have noticed that an across-stripe swap is a "vertical" slider swap).

Algorithm

Cells are assigned columnwise, top to bottom, unless indicated as exceptions below:

Step 1 - Assigning processor N (see figure 19), the non-overflow case.

```
while (the number of unassigned cells >= area)
    {
       - assign the cells for the processor columnwise top
         to bottom.
    }
```

Now assume that in figure 19, the unassigned area between columns j and s, inclusive, in stripe i, is not large enough to accommodate another complete component with A cells and thus is designated as the U-turn region.

The next processor to be assigned will overflow into the next stripe.

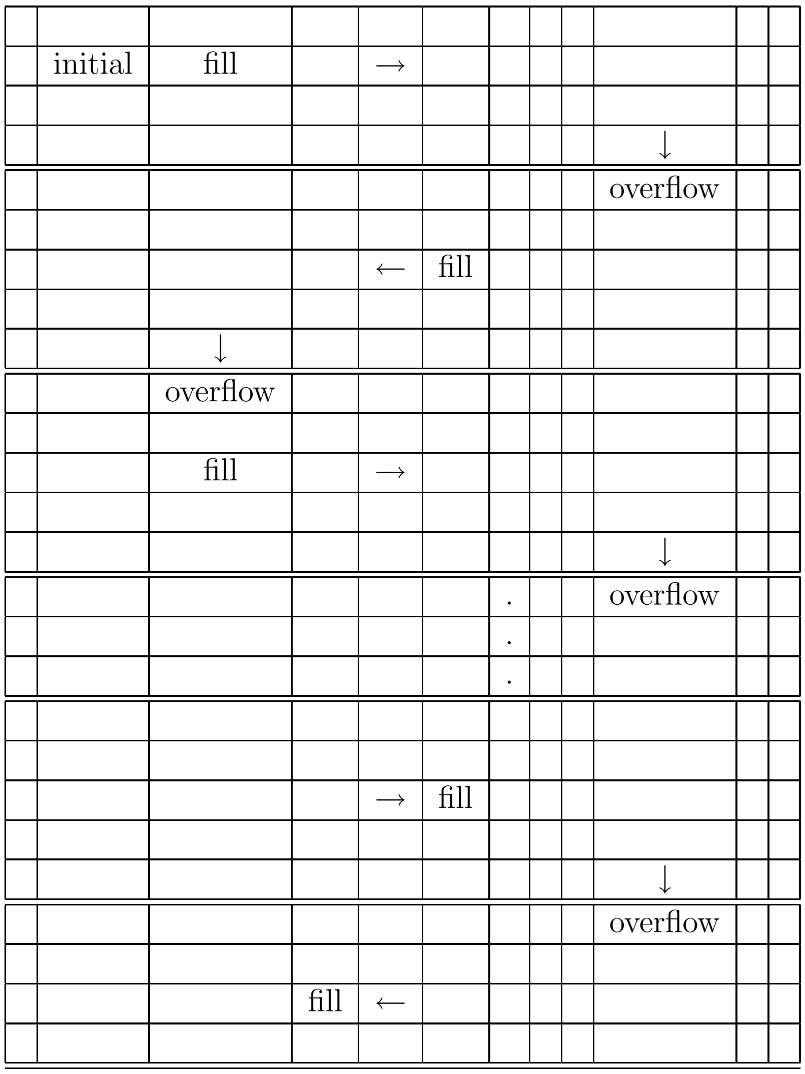


Figure 16: Flow of assignments

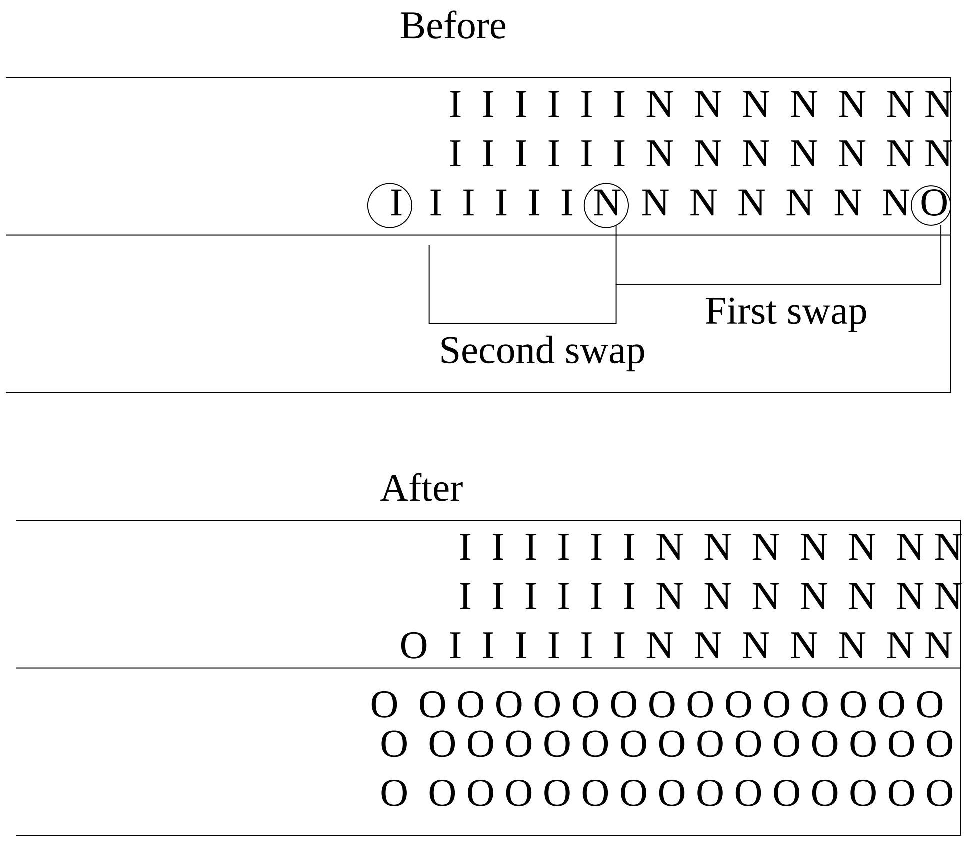

Before
I I I I I I N N N N N N N
I I I I I I N N N N N N N
I I I I I I N N N N N N N O
First swap
Second swap
After
I I I I I I N N N N N N N
I I I I I I N N N N N N N
O I I I I I I N N N N N N N
O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O
O O O O O O O O O O O O O O O

Figure 17: An example of a pair of slider swaps

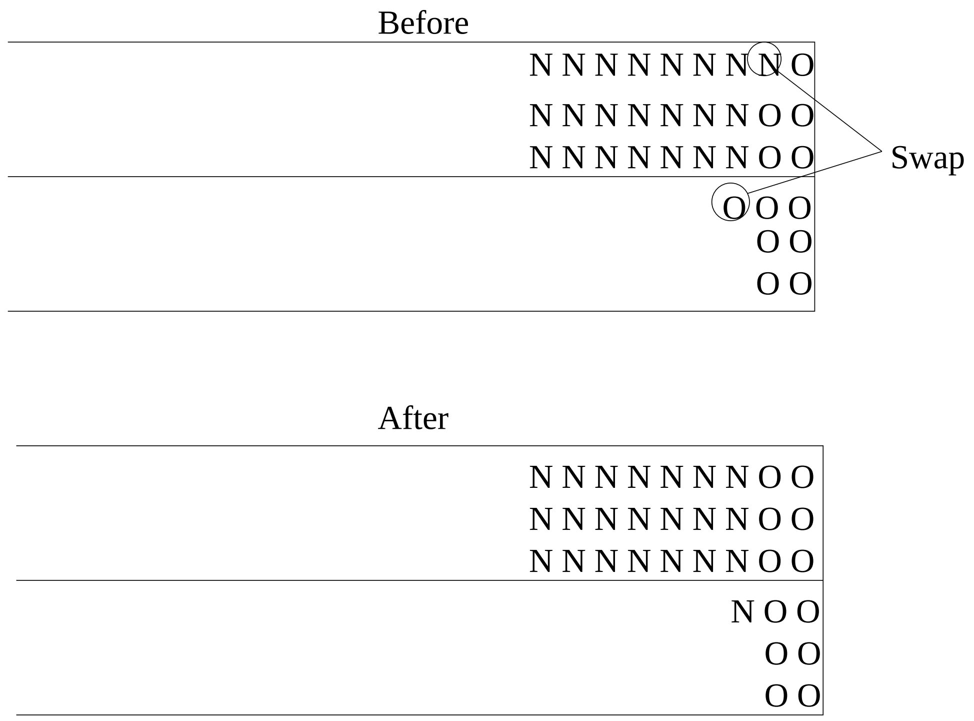


Figure 18: An example of an across-stripe swap

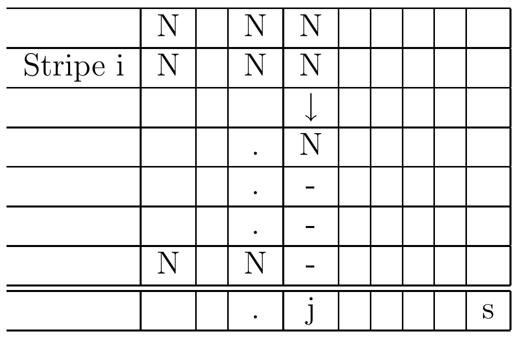


Figure 19: Assigning non-overflow component N

Step 2 - Assigning component O (see figure 20), the overflow case.

	N		N	N	O	O			O
Stripe i	N		N	N	O				O
				↓					.
			.	N					.
			.	O		O			
			.	O					
	N		N	O					O
column labels									s
	-	-	-	-	-	-		O	O
Stripe i+1	-	-	-	-	-	-		↓	↓

Figure 20: Assigning overflow component O

```
Assign all the remaining cells in stripe i to O.


while (there remain O's to assign)
{
   Assign the O's top to bottom in stripe i+1.
}


if (column s, in both stripes, is not completely assigned
    to O's)
{ // Balance heights in column s-1 and s via reducing swaps
  // (see figure 20).


     for the stripe in which the O's don't completely fill
      column s
        while (height of O's in column s-1) + 1 <
             (height of O's in column s)
```

```
          {
            swap the spike O with the lowest (highest) N (N', if the
            peninsula is in stripe i+1) in column s - 1.
          }
     // Taking care of the extra cell.
          if (height of O's in column s > height of O's in column s-1)
          {
           if (stripe == i)
           {
             assign this extra cell in the last row of stripe i in
             column s-2.
           }
           else
           {
             assign this extra cell in the first row of stripe i+1
             in column s-2.
           }
         }
  }
```

(Note: In figure 21, because of assumption 1 (stated below), the O in column s-2 can never be a spike. See lemma 3.9 for details.)

Step 3 - Making reducing swaps.

At this point, the grid has been completely assigned. We refer to this assignment as the **initial assignment**. For any pair of stripes, there can be at most one component

	N	N	N	N	N
	N	N	N	N	**O**
	N	N			
Stripe i	N	N			.
	N	N		.	
	N	N		.	
Z	N	N	N	N	**O**
					s

	N	N	N	N	N
	N	N	N	N	N
	N	N	N	N	N
	N	N	N	**O**	**O**
	N	N	N	**O**	**O**
	N	N	N	**O**	**O**
Z	N	N	**O**	**O**	**O**
					s

Figure 21: Removing a peninsula via swaps

that appears in both stripes. There are two types of swaps that can improve the total perimeter. The first is an across-stripe swap. The second is a slider multi-swap (referred to below as simply a slider). A slider occurs when a component from stripe i+1 (i) has a single cell in stripe i (i+1). Multi-swaps may involve several two-cell swaps. After performing these swaps in Step 3, the assignment is locally optimal.

Step 3 - Making reducing swaps.

```
i = 1;
  while ( i <= number of stripes - 1)
    {
```

Step 4.1 - Check for **across-stripe swap** for components adjacent to overflow components. Components assigned immediately after an O component are designated N' For the following block of code, refer to figures 22 and and figure 23.

N	N	N	N	N	**N**	O
N	N	N	N	N	O	O
N	N	N	N	N	O	O
N	N	N	N	N	O	O
N'	N'	N'	**O**	O	O	O
N'	N'	N'	N'	O	O	O

N	N	N	N	N	**O**	O
N	N	N	N	N	O	O
N	N	N	N	N	O	O
N	N	N	N	N	O	O
N'	N'	N'	**N**	O	O	O
N'	N'	N'	N'	O	O	O

Figure 22: An N-O across-stripe swap

N	N	N	N	O	O	O
N	N	N	**O**	O	O	O
N’	N’	N’	N’	N’	O	O
I	N’	N’	N’	N’	O	O
I	N’	N’	N’	N’	O	O
I	N’	N’	N’	N’	**N’**	O
					m	

N	N	N	N	O	O	O
N	N	N	**N’**	O	O	O
N’	N’	N’	N’	N’	O	O
I	N’	N’	N’	N’	O	O
I	N’	N’	N’	N’	O	O
I	N’	N’	N’	N’	**O**	O
					m	

Figure 23: A O-N’ across-stripe swap

```
    if (O has a side spike in stripe i(i+1)) &&
(O’s spike falls within the columns containing N) &&
        (N(N’) has a rightside spike in stripe i+1(i))
    {
         - swap the two spikes.
    }
```

At this point, N,O or N’ could have cells in both stripe i and i+1, but this is not true for any other component appearing in either stripe.

Observe that across-stripe swaps for O result in full height rectangles in stripes i and i+1 (and no O spikes), because these swaps have corner cells as destinations.

The last improving swap within stripe and is called a **slider swap**. This definition will be demonstrated by an example, see figures 24 and 25 (Bottom slider swaps are also possible, and are similar, hence are not illustrated here.).

stripe i	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2
stripe i+1	5	5	5	5	**4**	4	4	4	4	**3**	3	3	3	3	3	**2**
	6	5	5	5	5	4	4	4	4	4	3	3	3	3	3	3
	6	5	5	5	5	4	4	4	4	4	3	3	3	3	3	3
	6	5	5	5	5	4	4	4	4	4	3	3	3	3	3	3
	6	5	5	5	5	4	4	4	4	4	3	3	3	3	3	3

Figure 24: Before a top slider (the bold 2 is the slider)

stripe i	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2
stripe i+1	5	5	5	5	**2**	4	4	4	4	**4**	3	3	3	3	3	**3**
	6	5	5	5	5	4	4	4	4	4	3	3	3	3	3	3
	6	5	5	5	5	4	4	4	4	4	3	3	3	3	3	3
	6	5	5	5	5	4	4	4	4	4	3	3	3	3	3	3
	6	5	5	5	5	4	4	4	4	4	3	3	3	3	3	3

Figure 25: After a two top *slider* swaps

Step 4.2 - Check for a slider.

```
Search for a slider cell within each stripe.


Slide this cell by swapping as necessary.

(Several swaps may be required. )


}// This ends the while ( i <= number of stripes - 1).loop
```

At this point, all of the components in stripe i have locally optimal perimeters.

Proof of Local Optimality

Assumptions

1. Area for each processor $>$ 4*(smallest stripe height).
2. Stripe height ≥ 4.
3. Each stripe has at least 5 components.

Fact 3.4 *A swap is only made if it reduces the overall perimeter.*

We group the components into three types. The first, Type I (I is for interior), is a component that is not a neighbor of an overflow component in the same stripe, at the end of the initial assignment. The second type, Type O (O is for overflow), is a component that does extend over two stripes at the end of the initial assignment (these types of components will always span two stripes). The last component, Type N (N is for neighbor), is a component that is a within-stripe neighbor to a Type O component.

To prove local optimality, we will show that no component can be improved via a swap. Each type of component will be considered separately. When considering possible swap cells, only destinations with assignments that match neighbors of spike cells need be considered, since otherwise an island cell would result. We need the following definition:

Definition - For a given U-turn region, define those cells that are assigned to type-N components, type-O components, and any other component that is involved in a slider swap to be the **swap area** for the U-turn region.

We first need to prove that processing a U-turn region doesn't result in a configuration that would allow a chain reaction of other swaps. In order to prove this result, we need the following lemma and associated definition.

Definition - A component in which a slider cell was detected will be termed a **slider component**.

Lemma 3.7 *There is at least one component in each stripe that cannot participate in a slider swap.*

Proof (by contradiction) - A slider component can either be a N-component or an O-component. For this argument, assume that the slider components are N's (the argument for other cases is similar).

Assume that N and N' (see figure 26) are slider components. Because of the assumption about each stripe containing at least five components, it follows that neither N and N' can extend to the halfway column of the grid. From this it also follows there there exists a component I_s that neither component can affect. Also, N (N') can't affect anything to the right (left) of I_s. So it follows that two slider components cannot affect common components.

By a similar argument, if N and N' appear in consecutive stripes, then N (or N') cannot affect N' (or N) (see figure 26).

We have now shown that both stripes within a swap area area cannot be affected by any component outside the swap area.

□

This previous lemma is useful in that it shows that there is a termination point to the slider swap. We also know that a N-component could never involve another N-component from another U-turn region.

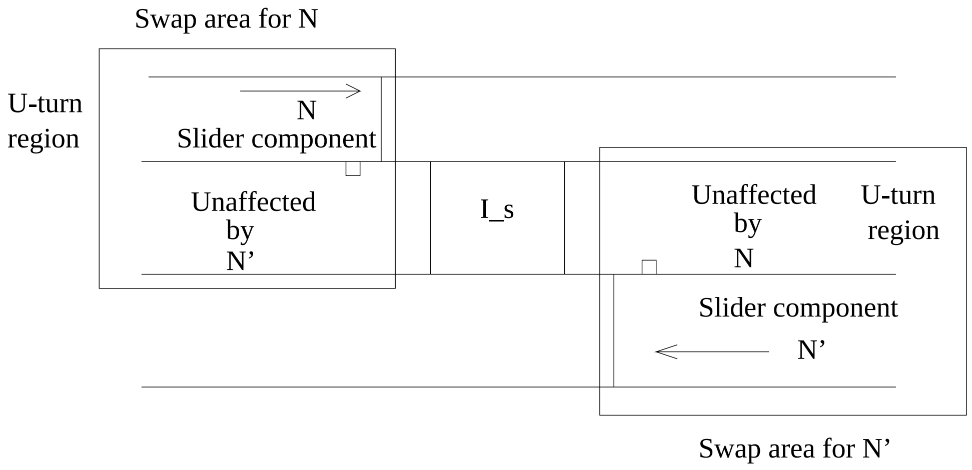


Figure 26: Component I_s can't be affected by slider swaps.

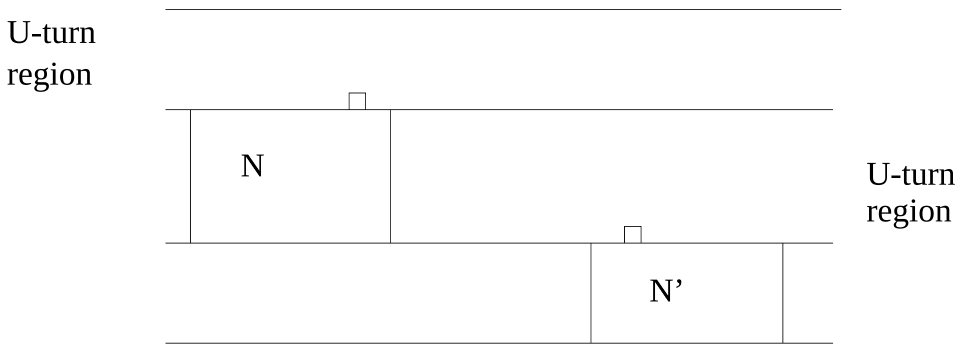


Figure 27: One slider component cannot affect another slider component.

This previous lemma is useful in that it shows that there is a termination point to the slider swap. We also know that a N-component could never involve another N-component from another U-turn region.

Lemma 3.8 *The final total perimeter for the domain is independent of the order in which the U-turn regions are processed.*

Proof - From the previous lemma, we know that a slider swap in one region does not change the assignment for a component in another swap area. The only other type of swap is an across-stripe swap, which involves only O and either N or N' component within the U-turn region. It follows that the swap areas are mutually exclusive. Therefore, we can process the U-turn regions in any order, without affecting the final solution.

□

Because of this lemma, we may processes the U-turn regions starting at the top of the grid and working down, as was done in the algorithm.

It is important that the reader understand the necessity of the previous two lemmas. The proofs that follow depend on potential spike positions being easily identified.

For the following proof, the direction of assignment is assumed to be left to right in the stripe that could produce an overflow into the next stripe. The argument is similar for the case of right-to-left assignment.

To prove the theorem, we must show that for each type of configuration that the configuration cannot be improved whether or not the configuration had been involved in some type of swap. Since the O-type configuration is where the swapping begins, this is the configuration that will be dealt with first. First we need to state some facts concerning slider swaps.

Facts concerning a slider swap

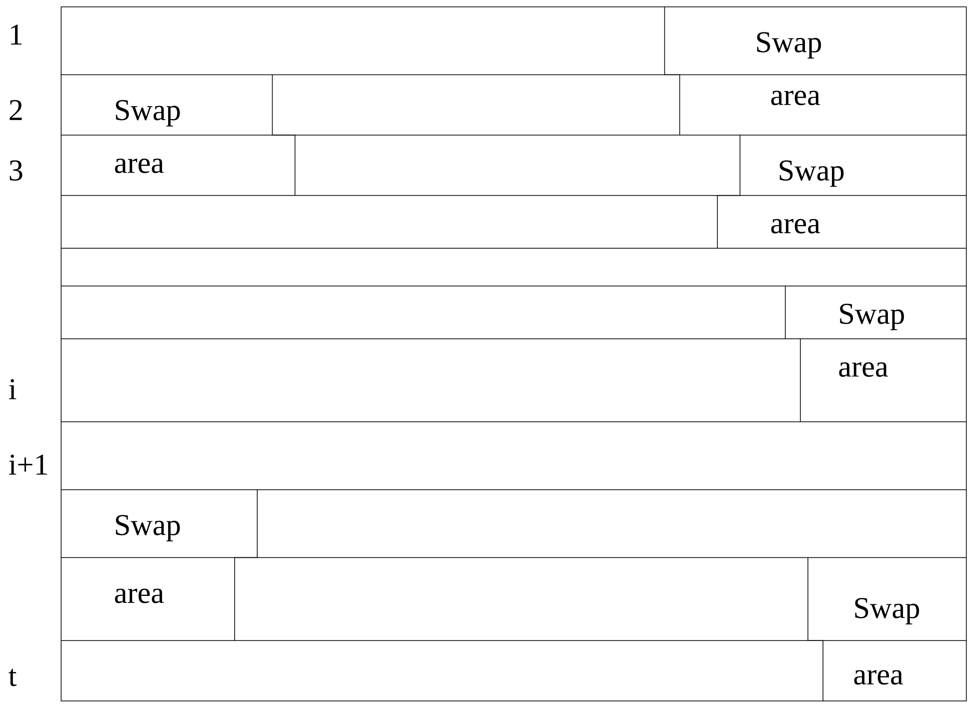


Figure 28: Mutually exclusive swap areas

1. The overflow for the slider component can only contain a single cell (which forms a spike).
2. Any component that is improved as a result a slider will have a rectangle for its final configuration.
3. A slider allows for across-component swaps.

Before we examine the O-component, we need the following lemma, which eliminates possible spike positions.

Lemma 3.9 *If the peninsula elimination procedure is applied to a component, then the number of columns completely assigned to that component in the other stripe is at least three.*

Proof - By assumption 1, we know that component O has enough area to be assigned to at least four columns, within one stripe. Since the peninsula elimination phase only occurs if the peninsula does not occupy a full column (and of height > 1), that forces at least three columns in the other stripe to be completely assigned to component O.

□

It follows that the odd O cell moved to column s-2 can never be a spike cell (see figure 30).

TYPE-O COMPONENTS

The case tree for the O component is shown in figure 29.

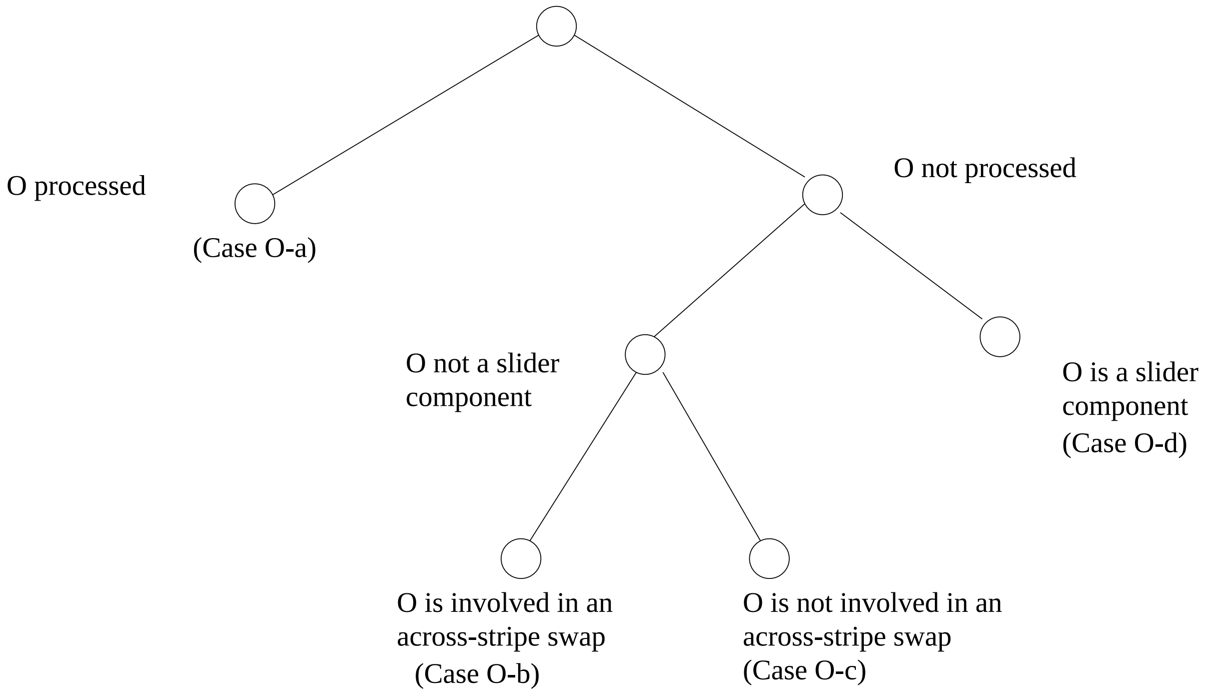

O processed
(Case O-a)
O not processed
O not a slider component
O is a slider component (Case O-d)
O is involved in an across-stripe swap (Case O-b)
O is not involved in an across-stripe swap (Case O-c)

Figure 29: Case tree for Type-O components

If O is processed (columns s and s-1 are assigned an equal number of O's), then O can't be a slider, because there is more than one O assigned in either stripe. Also, O can't be involved in an across-stripe swap, because neither N nor N' will have a spike in column s-1 (see figures 22 and 23).

If O has not been processed, then O could be involved in either an across-stripe swap or a slider swap (as only a slider component, since no other component can surround

O) but not both. If O has only a single cell in either stripe, then O could be a slider component. If there exists an across-stripe swap, then O has more than one cell in both stripes and cannot be a slider component.

O component is processed (Case = O-a in tree)

Here the O's were be reassigned to columns s-1 and s-2. To prove local optimality, we need only look at neighbors of spikes.

By lemma 3.9, the cell marked with an X is the only position that could contain a spike (this follows from the fact that at most three columns in the upper stripe will contain O's and by the lemma, at least three columns in the lower stripe contain O's, therefore there can be no spikes in the upper stripe). By lemma 3.4, if O's perimeter is to be reduced, then spike X must be moved to a corner position. At most there are two corner positions: 4_1 and 4_2 (there is only one, 4_1, if the O's only appear in a single row in stripe i). To move a 4 from either 4_1 or 4_2 would add a row to 4's height (since neither 4_1 or 4_2 is a spike cell), offsetting any improvement in O. Therefore, there is no improving swap.

Direction			4	4	4	4
$\rightarrow$			4	4	4	4
			4	4	4	4
			4	4	4	4
		4	4	4_2	O	O
stripe i	**4**	4	4_1	O	O	O
stripe i+1		**X**	O	O	O	O
		5	O	O	O	O
$\leftarrow$		5	O	O	O	O
		5	O	O	O	O
		q				s

Figure 30: Spike positions when O is processed out

The reader should note that a single-column peninsula is possible. However, if O completely fills the last column of a stripe, then this component is at a local minimum, with respect to two-cell swaps (Since the height of the stripe is at least four, this O peninsula could be folded in and reduce O's height by at least two, while increasing N width by 1. But this would require an initial non-improving two-cell swap, which has been disallowed.).

Direction	2	O	O	O	O
→	2	O	O	O	O
	2	O	O	O	O
stripe i	**W**	O	O	O	O
stripe i+1	5	5	5	O	O
	5	5	5	O	O
←	5	5	5	5	O
	5	5	5	5	**Z**
			q	q+1	s

4	4	4	4	**Y**
4	4	4	4	O
4	4	4	O	O
4	4	4	O	O
X	O	O	O	O
	O	O	O	O
	O	O	O	O
	O	O	O	O
		q	q+1	s

Figure 31: Possible spike positions for an O-component (column s is completely assigned to O).

Direction		4	4	4	4	**4**	**Y**
		4	4	4	4	O	O
Stripe i		4	4	4	4	O	O
		4	4	4	4	O	O
Stripe i + 1	**X**	O	O	O	O	O	O
		O	O	O	O	O	O
		O	O	O	O	O	O
		O	O	O	O	O	O
						q+1	s

Figure 32: When an across-stripe swap may not be made, even though there is a spike in column q+1. The O spike can also appear in the other stripe.

O is processed

No across-stripe swap was possible (Case = O-b in tree).

See figures 31 and 32, as fill proceeds, if W starts as a spike, we will either stop short of W's column when assign O's and W remains a spike, or continue filling with O's, past W's column, and X could possible become a spike. W (X) can only be moved to a corner by swapping across stripe. Since we know that the 5 (4) component does not have any cells in stripe i (i+1), this swap will increase 5's (4's) height, without reducing the width (the 5 (4) component is assigned completely to three columns and the far right column must contain at least two of this components cells **OR** (see figure 32) O's spike falls outside the columns containing N, which would have created a "4" island cell, if 4 and O had been swapped; otherwise an across-stripe swap would have been made).

To move Z (Y) to a corner position would require a 5 (4) to be assigned to column s. This increases the number of columns component 5 (4) appears in. And since 5 (4) appears in at least three full columns, the row count of component 5 (4) can't be reduced. Therefore, no improvement is possible.

A N-O across-stripe swap (Case = O-c in tree)

(See figures 22 and 23)

N has a side-spike cell in stripe i+1 (i) and O has a side-spike cell in stripe i (i+1). When these two cells are swapped, the N cell has effectively been slid from side to top or bottom of the component. After the swap, O will have no spike cells, but rather a rectangle in stripe i and a rectangle in stripe i+1 (each rectangle is of width at least 2, by assumption 1). It follows that O is at a local minimum.

Slider Swaps were performed

The O component is the slider component (Case = O-d in case tree)

(see figure 33)

In figure 33, both W and Z are potential O spikes. Since component O is a slider component, then Z is on the bottom border of the component. In this case, O is the only

	N	N	O	O	O	O	O
Direction	N	N	O	O	O	O	O
→	N	N	O	O	O	O	O
	N	**W**	O	O	O	O	O
		5	5	**Z**	4	4	4
		5	5	5	4	4	4
		5	5	5	4	4	4
		5	5	5	4	4	4
		b					

Figure 33: A top-slider swap component

component that appears in both stripes. The only corners to which Z could be moved are all in the upper stripe. But those positions are assigned to components that only appear in the upper stripe. So any reduction in Z's height will produce an increase in height for the swapping component.

For the case of W, we need only look at at components N and 5. The only corner position that W could be moved into occurs in the bottom stripe. Moving a 5 to the upper stripe will not decease 5's width, but will increase 5's height. Therefore, this type of swap produces no improvement.

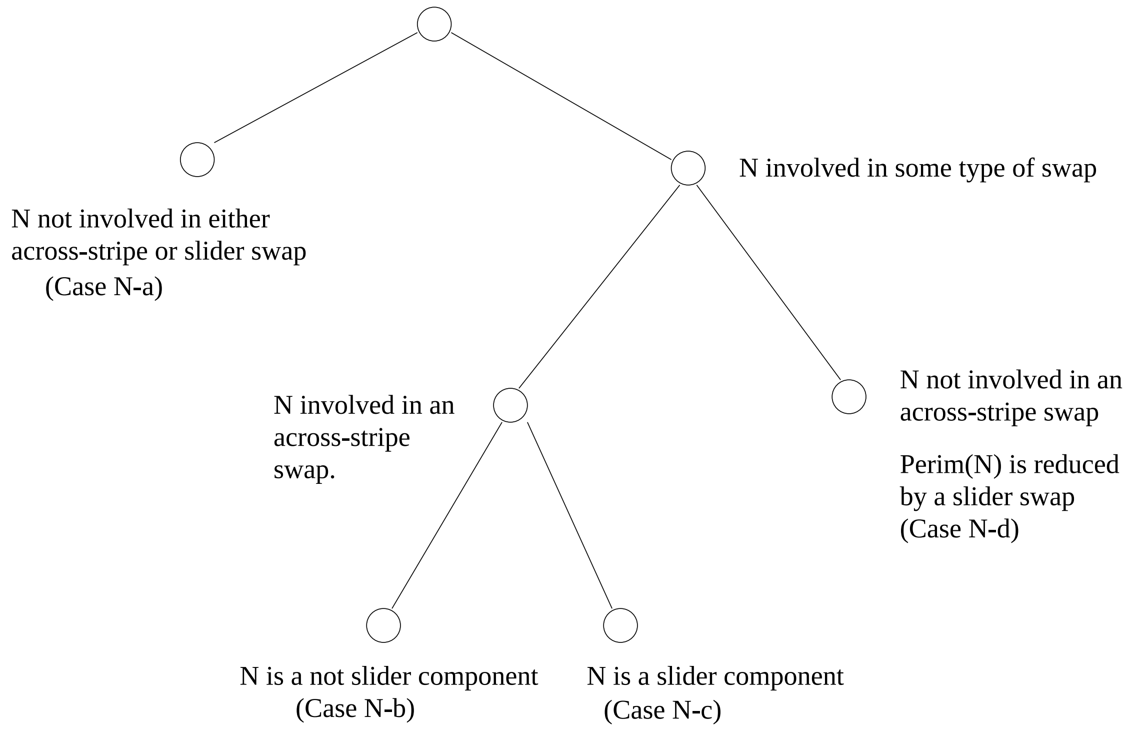
Therefore, this O component is at a local minimum before and after a slider swap. The argument for the slider cell (Z) being at the top of the component is similar.

TYPE-N COMPONENTS

The cases for showing that N is at a local minimum are listed in the tree as shown in figure 34. If N is not involved in any type of swap, then N only appears in one stripe, and only a within-stripe swap could possible improve N. In this case, N could not possibly be a slider configuration.

The other case occurs if N was involved in some type of swap. If N were involved in an across-stripe swap with O, then N may or may not be a slider component. If N were not involved in an across-stripe swap, then N appears only in a single stripe, but could be affected by a slider swap.

The arguments to follow apply to both N and N'.


N involved in some type of swap
N not involved in either across-stripe or slider swap (Case N-a)
N involved in an across-stripe swap.
N not involved in an across-stripe swap
Perim(N) is reduced by a slider swap (Case N-d)
N is a not slider component (Case N-b)
N is a slider component (Case N-c)

The case with N as a slider configuration can only occur after an across-stripe swap.
If Perim(N) is reduced by a slider configuration, then it cannot become a slider configuration.

Figure 34: Case tree for Type-N components

N not involved in any swaps (Case = N-a tree)

If the O component is not evened out in N's stripe, then in this case the type N component is assigned like a type I component. N can have at most two side spikes that cannot be moved within frame without making the perimeter for another component worse (This was shown to be true in the proof of local optimality of CM for the rectangular case.).

If the O component was processed in N's stripe, to reduce N's perimeter would cause O to become non-slice convex (see figures 21 and 30, where N spikes are boldfaced Z and 4). Since N occupies more than four columns and only possible spike position is at the left boundary, the N spike would have to be moved to a corner position, reducing N's perimeter by two. In doing so, we would make O non-slice convex, thereby increasing O's perimeter.

N involved in a swap

N involved in an across-stripe swap, but is not a slider component. (Case N-b in tree)

The side spike for N has been slid to either the top or bottom of the component. The N component can now only have one side spike and a slider spike. The bottom spike can occur in any cell between columns b and e, inclusive, as shown in figure 35 (Otherwise, the column count for N would increase and no improvement in total perimeter possible. This could only occur if O's spike is in a column outside of N's enclosing frame; and we said that this swap would not be made.).

I	N	N	N	N	N	O
I	N	N	N	N	N	O
I	N	N	N	N	N	O
N	N	N	N	N	N	O
b			**N**		e	O

Figure 35: N component after an across-stripe swap

As in the case for the O component, we have shown that the leftside N spike can't be moved within frame and that the bottom N spike can't be moved into the upper stripe in column b. The only other possible move for the N spike is to column e+1, but there are no corners in this column.

It follows that N's component, as the slider component, is at a local minimum after one or more slider swaps.

N is in an across-stripe swap AND is a slider component. (Case N-c in tree)

The proof is exactly the same as for case N-b. The spikes, at the side or bottom, can't be moved within the enclosing rectangle without increasing the perimeter for the other component.

N was subjected to a slider swap only (Case = N-d in tree)

This situation can occur if the neighboring O component in stripe i (i+1) is the slider component. In this case, N component in stripe i+1 (i) becomes a rectangle and cannot be improved.

TYPE-I COMPONENTS

Figure 36 is the case tree for proving that I is at a local minimum. From the discussions for the two previous types of components, we know that I is never involved in an across-stripe swap (swapping with either the O component or the N component would not produce an improved total perimeter). Therefore, an I-component can only appear in one stripe and can never be a slider component. A type-I component can only be involved in slider swaps as the component whose perimeter is going to be reduced. So we only have to look at the cases of slider swap or not.

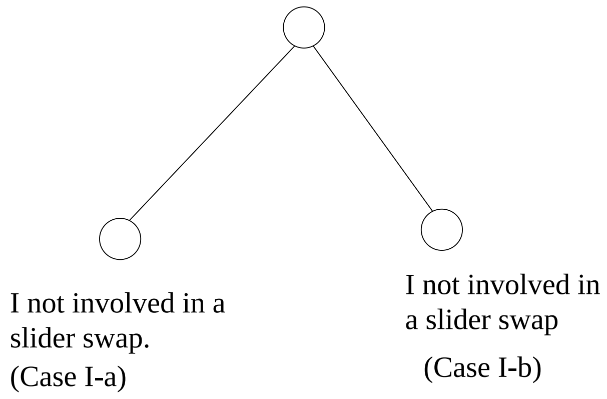

I not involved in a slider swap.
(Case I-a)
I not involved in a slider swap
(Case I-b)

Figure 36: Case tree for Type-I components

Fact 3.5 *For type I components, only within-stripe swaps offer the possibility of improvement.*

I not involved in a slider swap (Case = I-a)

Previous Stripe	C	C	C	C	C	C
Direction		**Y**	I	I	**C**	
←			I	I	I	
			I	I	I	
			I	I	I	
			I	I	I	
			D	I	I	**X**
	D	D	D	D	D	D

Figure 37: A Type I component

To swap cells with another type I processor will not reduce the perimeter (this was shown to be true in the proof of local optimality of CM for rectangular grids). A non-slider I-N swap is analyzed in the same way, and therefore no improvement is possible. An I-O swap within stripe is impossible, because I and O are not adjacent within the same stripe..

I involved in a slider swap (Case = I-b)

Case - An I-O/N swap (see figure 37)

I is a rectangle and cannot be further reduced (in fact, any swap will increase I's perimeter).

We have now shown that the grid is at a local minimum, by showing that none of the components can be improved after the second phase of the algorithm.

□

The next algorithm was developed to reduce the effect of peninsulas on the overall perimeter.

3.6 The Improved U-turn Algorithm

The Basic U-turn algorithm is an extension of the CM algorithm. Initially, only a single processor may extend over stripe boundaries. This algorithm has been proven to produce locally optimal solutions; although undesirable components can still occur. The size of the U-turn region is greater than or equal to zero and less than one component.

A further refinement was devised: The Improved U-turn algorithm. In this algorithm the U-turn region was expanded to be of size greater than one component and strictly less than two components (in the implementation the number of cells could exactly equal two components). Within this expanded region much more elaborate methods were used for assigning the components (We believe that each stripe must contain enough cells for at least seven components, in order to guarantee that neighboring U-turn regions don't overlap.). The motivation behind this algorithm is to reduce the possibility of getting components with large perimeters. The generic version of the Improved U-turn algorithm is:

1. Assign the C component row-wise in the upper stripe.
2. Assign the 3 component column-wise across both stripes.
3. Assign the remaining cells in the upper stripe to L's.
   Any remaining L's are assigned row-wise in the bottom
   stripe.
4. The M component is assigned column-wise in the bottom stripe.

One of the main differences is that last component that can be completely assigned within the stripe is assigned row-wise. Intuitively, this has the effect of reducing by half the height that an overflow component could have in stripe i. The Improved U-turn algorithm is like the Basic U-turn algorithm in that after an initial assignment, certain areas of the grid are searched for improving swaps. There is a trade-off that must be considered, however. A component assigned using the Basic U-turn algorithm in a near-optimal shape may be assigned in a shape with a perimeter far from optimal using the Improved U-turn algorithm. In figure 38, the component labeled C would have been assigned in a shape with a perimeter closer to optimal using the Basic U-turn algorithm. So a tradeoff takes place: making worse the perimeter of one or more components (the perimeters for L and M may also be made worse) in the hopes of improving the total perimeter for the components appearing in the U-turn region.

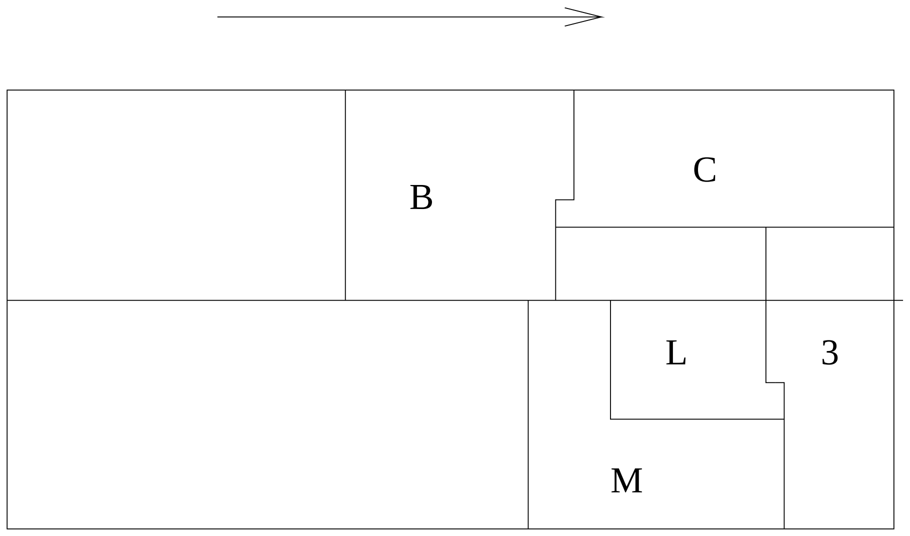


Figure 38: An Improved U-turn assignment

In related research, Donaldson and Meyer [Don97] present a full description of the Improved U-turn algorithm and a proof of local optimality (the number of cases is huge). The proof of optimality is similar to the proof for the Basic U-turn algorithm, except for a greater number of cases and refining swaps have to be checked.

There are cases that the Improved U-turn algorithm doesn't process well. This led to the development of nine different assigning techniques for a U-turn region. All of the techniques are based on a U-turn region of size greater than one component and less than two. The actual patterns of assignment are shown in chapter 6.

The theme of all these patterns is to use the best possible assigning pattern for a given situation. There are cases for which CM does a good job. There are other cases where column assigning does better. At each U-turn region, all nine assignments processes are tried and the best perimeter is kept.

There is another more subtle benefit that comes with the identification of a U-turn region. At the point where the U-turn region starts, an integral number of components will have been assigned. The perimeters of these completely assigned components are independent of the assignments for the rest of the grid. This is a key idea that will be exploited in the next chapter when defining subproblems.

3.7 Unbalanced Partitions

The methodology developed in this chapter is still valid if P does not divide the number of cells in the grid, $|V|$. Our arguments were not based on the area per component, but if there were enough remaining cells within a stripe to to assign a component. If P does not divide $|V|$, then ($|V|$ mod P) components will be assigned to (($|V|$ div P) + 1) cells. The remaining components will be assigned to ($|V|$ div P) cells. The components can be assigned in any order, just so long as a record is kept of how many of each type of component have be assigned.

3.8 Summary

Besides developing algorithms that produce locally optimal solutions, the identification of the U-turn area was the most important discovery of this section. For most stripes, those components appearing outside of the U-turn region tend to have perimeters that are close to the optimal perimeter. But bad things happen within the U-turn region. In the implementation that follows, several different stripe assignment procedures are used to reduce the chance of occurrence of components with perimeters that deviate greatly from the lower bound.

The second benefit occurred by careful consideration of the role of the U-turn region. The U-turn region is basically that part of a stripe that possibly contains a non-integral number of components. That part of the stripe that precedes the U-turn region plus all the cells appearing above the stripe can be assigned an integral number of components, in other words a subproblem. This fact plus the fact that an optimal solution possesses two traits that are present in problems that dynamic programming is applicable lead to the discovery of polynomial-time algorithms.

Chapter 4

Subproblems for the Dynamic-Programming Approach

4.1 Introduction

The ability to define a subproblem [DM99] is the foundation upon which the algorithms to be presented are based. The subproblems yield an efficient way of organizing intermediate results, which in turn eliminates redundant calculations. This reduction in redundancy allows an expanded set of graphs to be handled (as compared to Martin, who considered only the case of domains yielding stripes containing an integral number of components) and finds the optimal set of stripe heights (as compared to Christou-Meyer), while maintaining a polynomial run-time.

For a given set of stripes, the number of components varies from stripe to stripe. Within a stripe, after a component has been completely assigned, we know that an integral number of components have been assigned. This implies that a grid graph can be partitioned into two parts, each containing an integral number of components, by just splitting the graph at the last component assigned. Which component should be used for this division? By assumption, each stripe does contain a final component that is completely assigned to the stripe. (If this was the last component that could have been completely assigned within the stripe, then the Basic U-turn algorithm can be used for assigning the cells. If the next to last component was chosen, then the Improved U-turn Algorithm can be used.) This last component assigned also defines a U-turn region.

In this chapter, three main topics will be discussed. The first is subproblem definition. The second topic is the relationship between subproblems and stripes. The last point is that the subproblems possess properties that may be exploited in order to construct a shortest-path version of the problem and a dynamic-programming solution of the original problem.

4.2 Terms and Definitions

The following example, figure 39, is presented in order to demonstrate concepts and definitions that will appear in the paper.

For this grid, there are 360 cells to be assigned among 9 processors. The 26 rows (numbered 0 to 25) have been partitioned into four stripes. At most one component is allowed to "overflow" from one stripe into the next stripe. All components are assigned column-wise, top to bottom. For those components that appear in two stripes, all the remaining cells in stripe i are assigned first, then the remaining required number of cells are assigned column-wise in stripe i+1. The total perimeter of this solution is 294. In chapter 5, figure 52 contains a four-stripe optimal stripe partition of the same graph which produced a perimeter of 282. The fill procedure used for obtaining this optimal solution is described in Chapter 6.

Definition - A **begin-end row pair** defines a **stripe** as the collection of cells with corresponding row indices (in figure 39, stripe 1 is defined by the begin-end-row pair (0,6)).

Definition - A row in a grid is said to be an **end-row** if it is the last row in a stripe

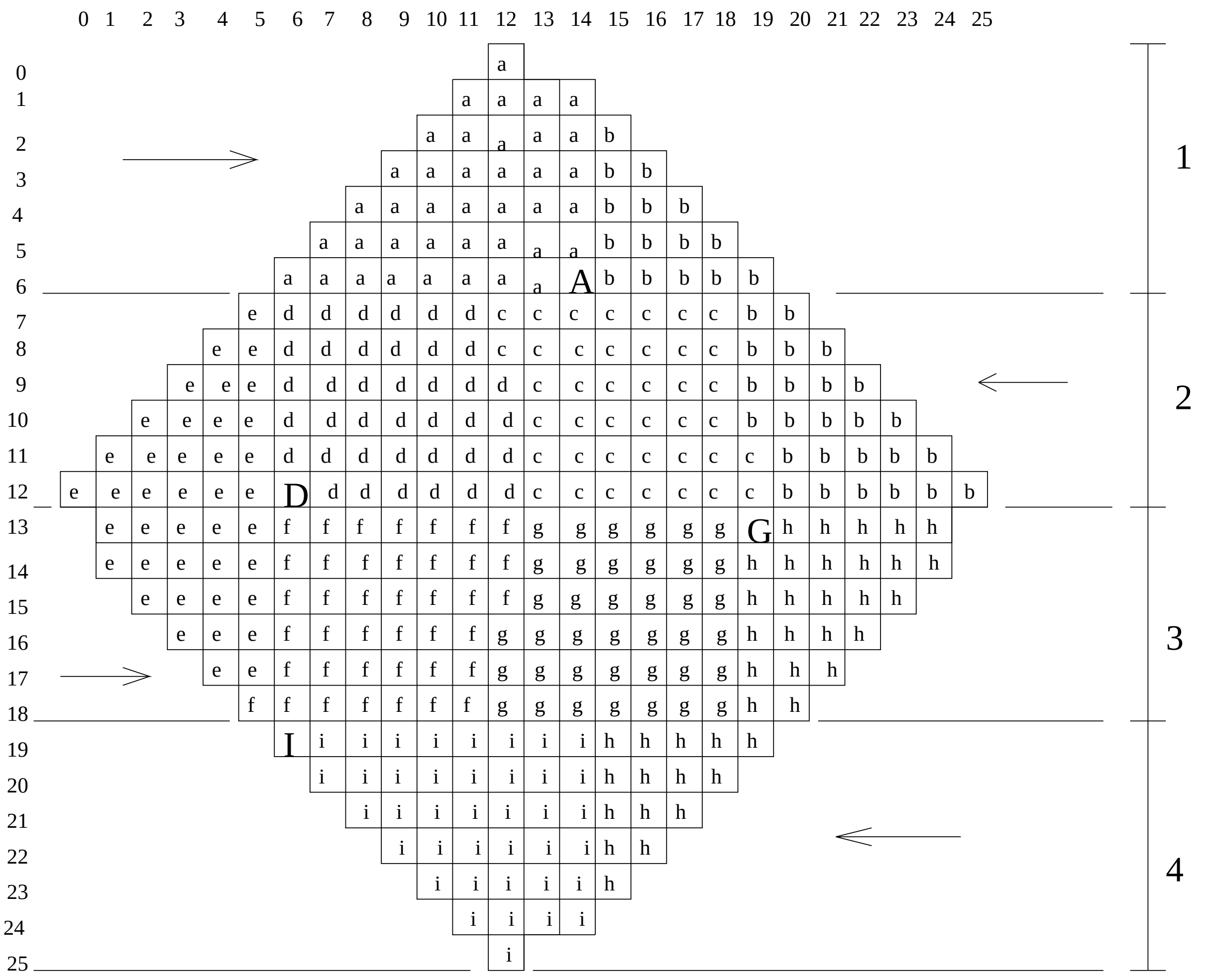


Figure 39: An assignment of a grid graph

(in figure 39, rows 6, 12, 18, and 25 are end-rows). A **begin-row** is defined similarly.

Definition - A subset of the grid is said to be **filled** if all of its cells have been assigned.

Definition - Within a stripe, a **divider** cell will be the last cell of the last component whose assignment was completed within a stripe (in 39, the uppercase A, D, G, and I are divider cells). The reader should understand that for the given example, the divider cells corresponded to the last components that could be assigned within the stripe. In the implementation, the divider cell was chosen to be the last cell assigned to the next-to-last component that could be completely assigned within the stripe (under most circumstances). In the implementation, the divider cell is a data structure that eliminates the need to keep track of individual cells in in the U-turn region (defined below).

For stripe i, defined by the begin-end row pair (b_i, e_i), we assume that a divider cell position for this stripe is uniquely determined by the fill procedure and the pair (b_i, e_i), and is independent of prior stripes. We assume that all cells in stripe *i-1* up to and including the divider cell are assigned before we consider the assignment of cells in stripe *i*. This will allow an incrementally discovered partition of the graph.

Definition - A **U-turn** region for stripe i consists of those cells within stripe i that "follow" the divider cell and hence are not assigned until stripe *i+1* is considered.

In figure 39, the upper regions corresponding to the cells assigned to components b, e, and h are the U-turn regions.

Definition - Those cells required to complete the assignments for components assigned within the U-turn region are called **overflow**.

Definition - A begin-end-row pair is **valid** if the corresponding stripe contains a divider cell.

Definition - A set of end-rows (r_1, r_2, ..., r_i) is said to be a **feasible sequence of end-rows** if:

for i $\geq$ 0, (r_i+1, r_{i+1}) is a valid stripe (where r_0 = -1).

The corresponding sequence of begin-end-row pairs is ((0,r_1), (r_1+1,r_2),..., (r_{i-1}+1,r_i)) (note: the use of valid-stripe sequences implies the existence of a divider cell within each stripe).

The expressions component and configuration will be used interchangeably below. Although, in the graph literature components are normally considered to connected, we do not assume this below.

Definition - A subproblem is **valid** if and only if it can be partitioned by a feasible sequence of end-rows.

The next definitions applies to the "state graph" constructed in the shortest-path version of the stripe-height problem.

Definition - A vertex in the state graph is uniquely defined by a begin row, an end row, and direction of assignment within that stripe in the original grid graph. In the figures to follow, a vertex is defined by (b_i, e_i, d_i). An ordered triple may be a valid vertex label if and only if b_i and e_i define a valid stripe.

For the constructions to follow, we further restrict what vertices will be considered. We only consider vertices that correspond to valid subproblems.

For a given pair of vertices, (B,E,D) and (B',E',D'), there is a directed edge connecting the vertices if and only if D = (D + 1) mod 2 and B' = E + 1.

In section 4, for the original grid, it will be shown that

1) b_i, e_i, and d_i uniquely define a subproblem;

2) If there is an edge between (B,E,D) and (B',E',D'), then (B',E',D') is a one-stripe extension of the subproblem defined by (B,E,D); and

3) The cells contained within this one-stripe extension are uniquely defined by (B,E,D) and (B'E',D').

Definition - Define perim (b_{i-1},b_i,e_i,d_i) to be the perimeter for the cells falling within the stripe defined by the extension of (b_{i-1},e_{i-1},d_{i-1}) to (b_i,e_i,d_{i-1}+1 mod 2), where $b_i = e_{i-1} + 1$.

4.3 Defining Subproblems

4.3.1 Identifying Subproblems

In this section, a method for breaking up the original problem into similar but smaller subproblems will be presented. **This is a the key step in the dynamic-programming and greedy algorithms that will be presented later in this dissertation**.

Definition - Given a grid G that contains M rows. Let b_i and e_i be two row indices within G such that $b_i \leq e_i$. Let d_i be the direction of assignment within this stripe. And let c_i be a divider cell appearing within this stripe. We say that b_i, e_i, and d_i define a **subproblem**.

In the following example, we assume that an integral number of components can be assigned to the cells in the union of regions A and B, where c_i denotes the divider cell in stripe i.

If the graph is symmetric, then b_i and e_i uniquely define a subproblem. If the grid is not symmetric, then direction of assignment becomes a factor.

For the two configurations appearing in figure 41, the assignments for the last stripes

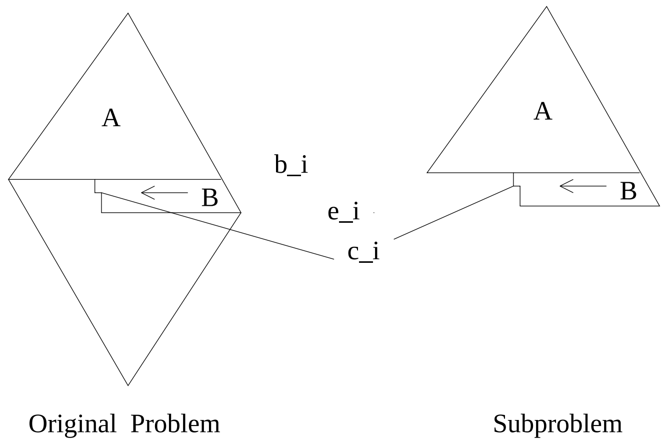


Figure 40: Subproblem example

are in opposite directions. Both subproblems are defined by the same pair of rows, but the configurations (U and U') of remaining unassigned cells are different. Again, this difference may result in different perimeter increases when the next stripe is added.

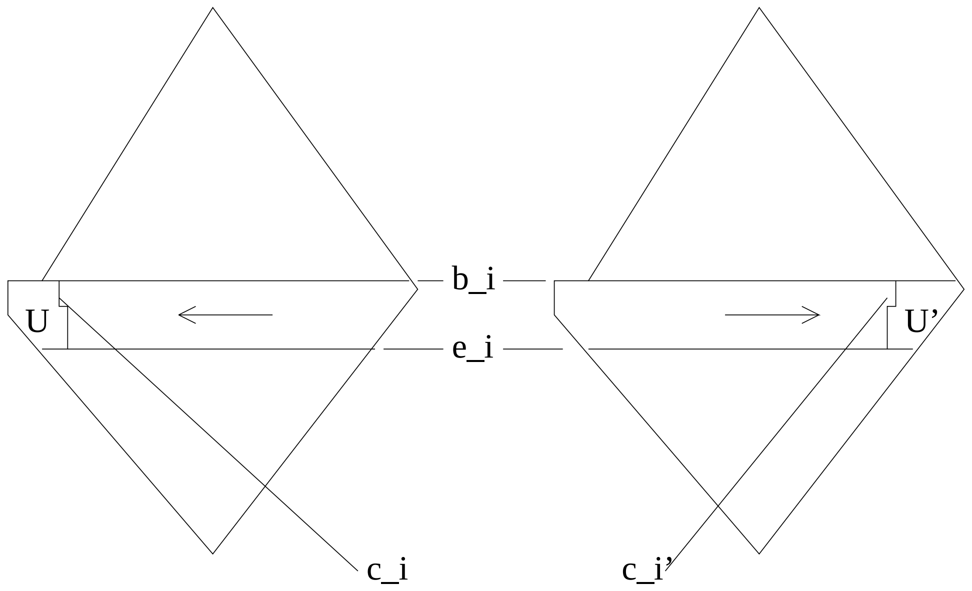


Figure 41: Direction of assignment affects the configuration of remaining cells.

For the discussion to follow, we assume that each stripe contains a divider cell and focus on the role of the divider cell within a stripe.

4.3.2 Selecting a Divider Cell

For the arguments to follow, the choice of divider cell c_k for a given stripe (b_k,e_k,d_k) must be independent of the height of the previous stripe. Also, for a given striping assignment, once a divider cell has been identified, the configuration of unassigned cells that appear within the given stripe (U-turn region) must be independent of the height of the previous stripe. Any assignment of components to the cells of the stripe may be used provided that the U-turn region is independent of the previous stripe and all unassigned cells are in the current stripe (and none are in the previous stripe).

For the algorithms to follow, we choose c_i to be the last cell in the last component assigned (assuming all the cells in the previous stripe have been assigned and a column-wise assignment was used for the last components) before the U-turn region for stripe (b_i,e_i,d_i).

For a given fill procedure, a begin-end-row pair and the direction of assignment uniquely define a subproblem (by default, also uniquely defined are the U-turn region for the current stripe and the remaining cells that will have to be assigned). In particular, these three variables uniquely define the last cell of a subproblem. Assuming that the subproblem is valid, then figure 42 shows how a begin-row, end-row, and direction of assignment determine a subproblem, independent of how previous subproblems have been defined.

For our choice of c_i, we are guaranteed that all of the cells above stripe (b_i,e_i,d_i) will have been assigned before any cells in this stripe are assigned. In figure 42 the number of cells above row b_i is independent of the choice for b_{i-1}, assuming that there exists valid subproblem defined by (b_{i-1},b_i-1,d_i + 1 mod 2). It then follows that the U-turn region within the stripe defined by stripe (b_i,e_i,d_i) is independent of the previous stripe.

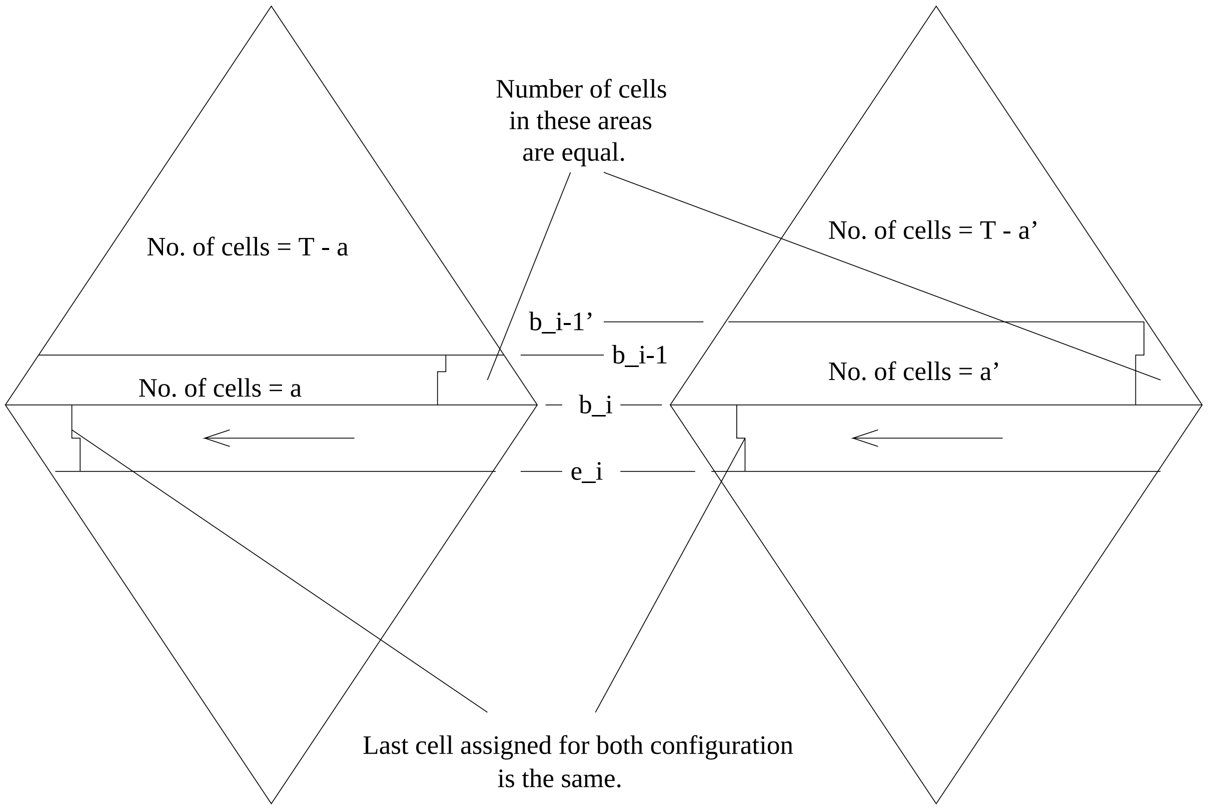


Figure 42: Begin-row end-row and direction uniquely determine a subproblem.

Subproblems:				Perimeters:			
No.	b_i	e_i	d_i	U-turn region(U_i)	Current stripe(S_i/U_i)	Total Increment	Subproblem Perimeter
1	0	6	1	0	32	32	32
2	7	12	0	44	54	98	130
3	13	18	1	34	56	90	220
4	19	25	0	42	32	74	294

Table 4: Dynamic-programming data for partition in figure 39

This deterministic way of uniquely defining subproblems allows for the use of dynamic-programming techniques for organizing subproblem results and eliminates redundant calculations (e.g. the subproblem defined by (b_i, e_i, d_i), and implicitly c_i, may be contained within several other larger subproblems). This ability to define subproblems also provides a useful way of identifying vertices and calculating edge weights in the corresponding shortest-path problem (see Chapter 5,"Description of a State Graph").

Consider the partitioned grid in figure 39. Examples of the two ways that the subproblem data can be presented are shown in table 4 and figure 43. Table 4 contains the perimeter data as organized by the dynamic-programming implementation. Figure 43 presents the data in the form of a weighted graph, with vertices corresponding to subproblems and edge weights equal to the perimeters for the incremental stripes.

The perimeter for the partition in figure 39 equals 294 (this is the final subproblem perimeter in the far right-hand column in table 4). This is not an optimal partition. An optimal stripe-based partition, with perimeter equal to 282, is shown in Chapter 5, figure 52.

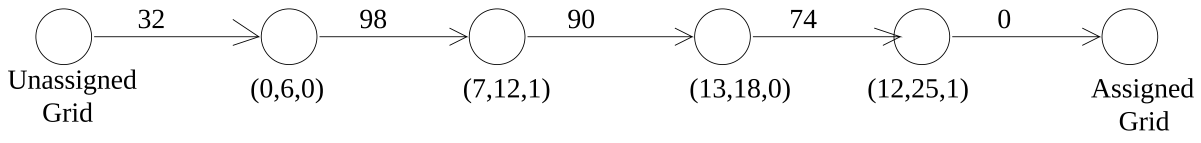


Figure 43: Path Data for table 4

4.4 Relationship between Subproblems and Stripes

A stripe is defined by a begin-end row pair and direction of assignment. After an assignment is made, there will be components completely assigned within the stripe plus possibly fractional parts of other components at both ends of the stripe (these components are assigned to cells appearing in more than one stripe). For this discussion, there can be at most two partial components appearing within the stripe. In particular, for stripe *i*, these partial components would correspond to the overflow from stripe *i-1* and the component assigned partially within the U-turn region for stripe *i*.

As was stated earlier, (b_{i-1},e_{i-1},d_{i-1}) uniquely defines a subproblem. These three quantities also define the U-turn region appearing within stripe *i-1*. By the same reasoning, (b_i,e_i,d_i) determines the U-turn region for stripe *i*. We now have a mechanism for defining the cells to be included within an "appending" (partial) stripe: those cells that would have to be appended to subproblem (b_{i-1},e_{i-1},d_{i-1}) in order to get subproblem (b_i,e_i,d_i) (note: the cells in the U-turn region for stripe *i* are not included).

Those cells appearing within this "appending" stripe may be assigned independently of all other cells. This follows from the fact that we have identified a group of cells that an integral number of processors can be assigned and that the number of cells defined by the subproblem (b_{i-1},e_{i-1},d_{i-1}) also allows for an integral number of components to be assigned.

A feasible solution of the problem can then be thought of as a sequence of "appending" stripes.

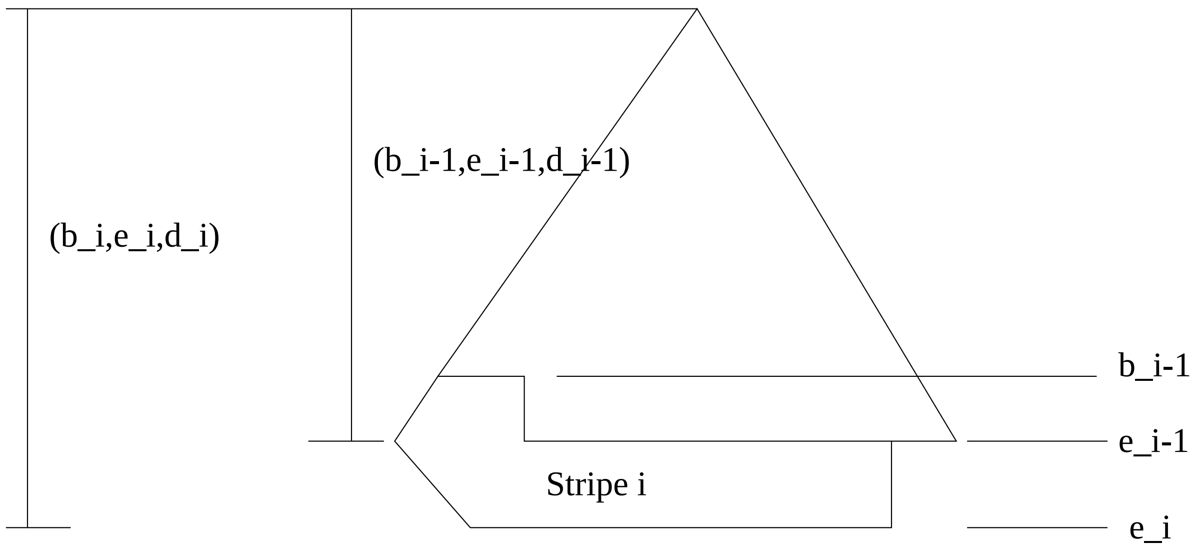


Figure 44: Two subproblems defining a stripe.

4.4.1 Construction of a Stripe-based Solution

Perimeter is computed in our approach incrementally, stripe-by-stripe as follows (see figure 45): perimeter (in the first stripe (0,e_1,d_1) - U_1) is computed, then the perimeter for (b_2,e_2,d_2) + U_1 - U_2, is added, etc. Figure 45 shows a fourth iteration of this process.

4.5 Properties of Subproblems

4.5.1 The Existence of Common Subproblems

Figure 46 shows two different sequences of stripe-heights used to partition the given grid. Both cases contain the subproblem ending with the stripe (b_5,e_5,d_5).

When the same subproblem appears in several other larger problems, the overall problem is said to possess the property of common subproblems. With respect to the graph viewpoint, a subproblem is a node and this node appears in multiple paths.

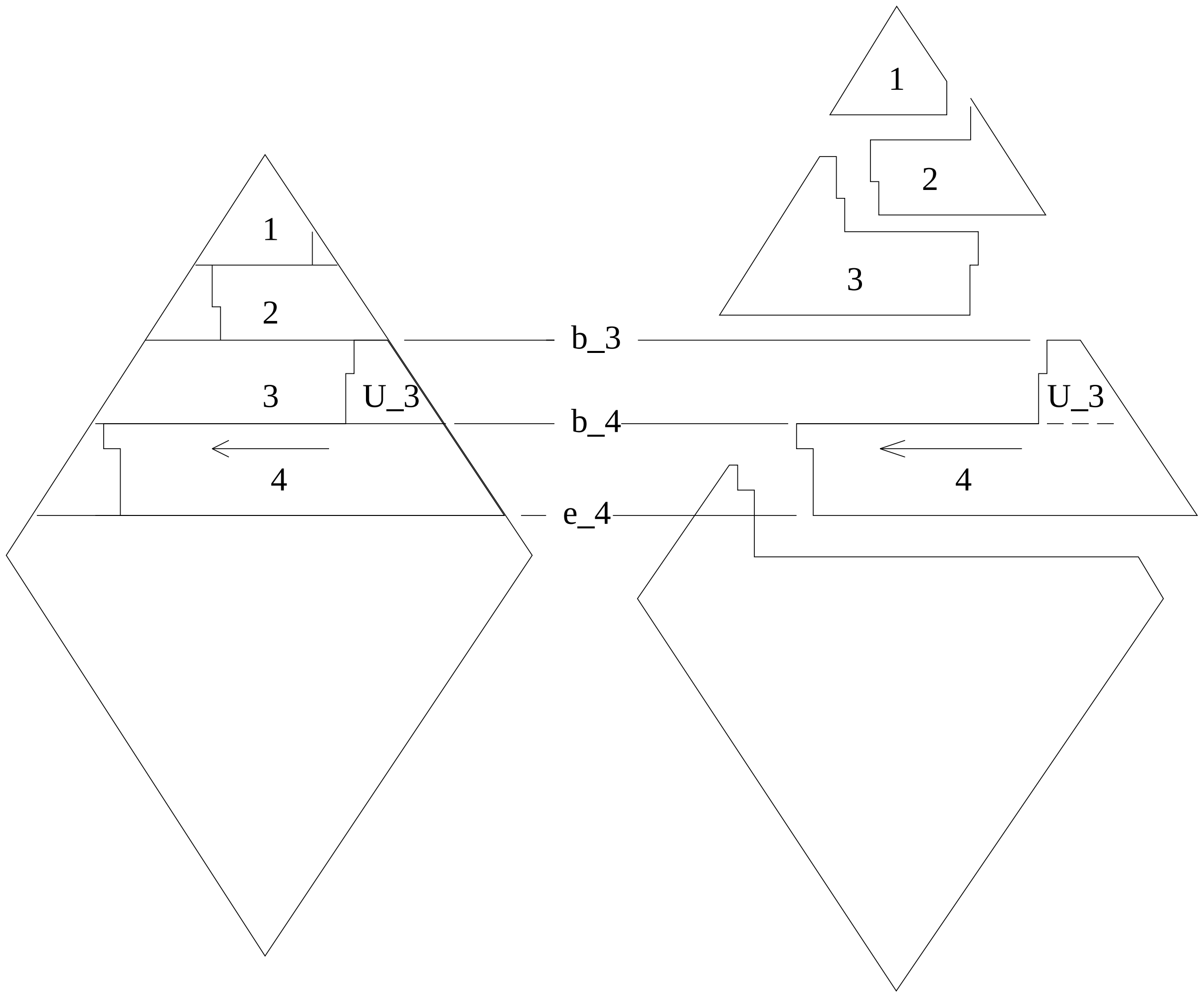


Figure 45: Incrementally processing a grid

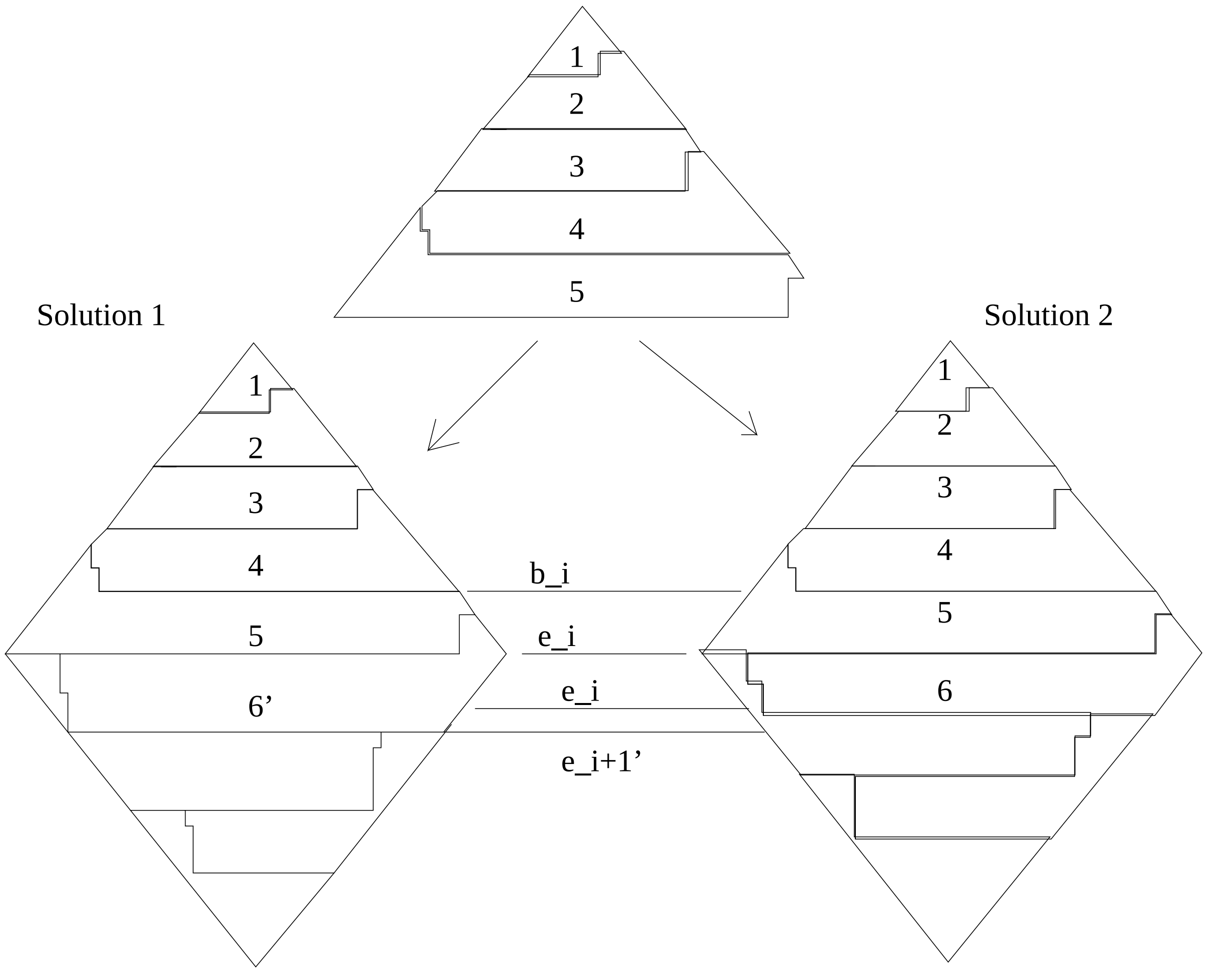


Figure 46: Feasible solutions with the same subproblem (stripes 1-5).

4.5.2 Optimal Substructure within a Solution

Cormen, Leiserson, and Rivest [CLR90] state that a problem exhibits "optimal" substructure if an optimal solution contains the optimal solutions for the corresponding subproblems.

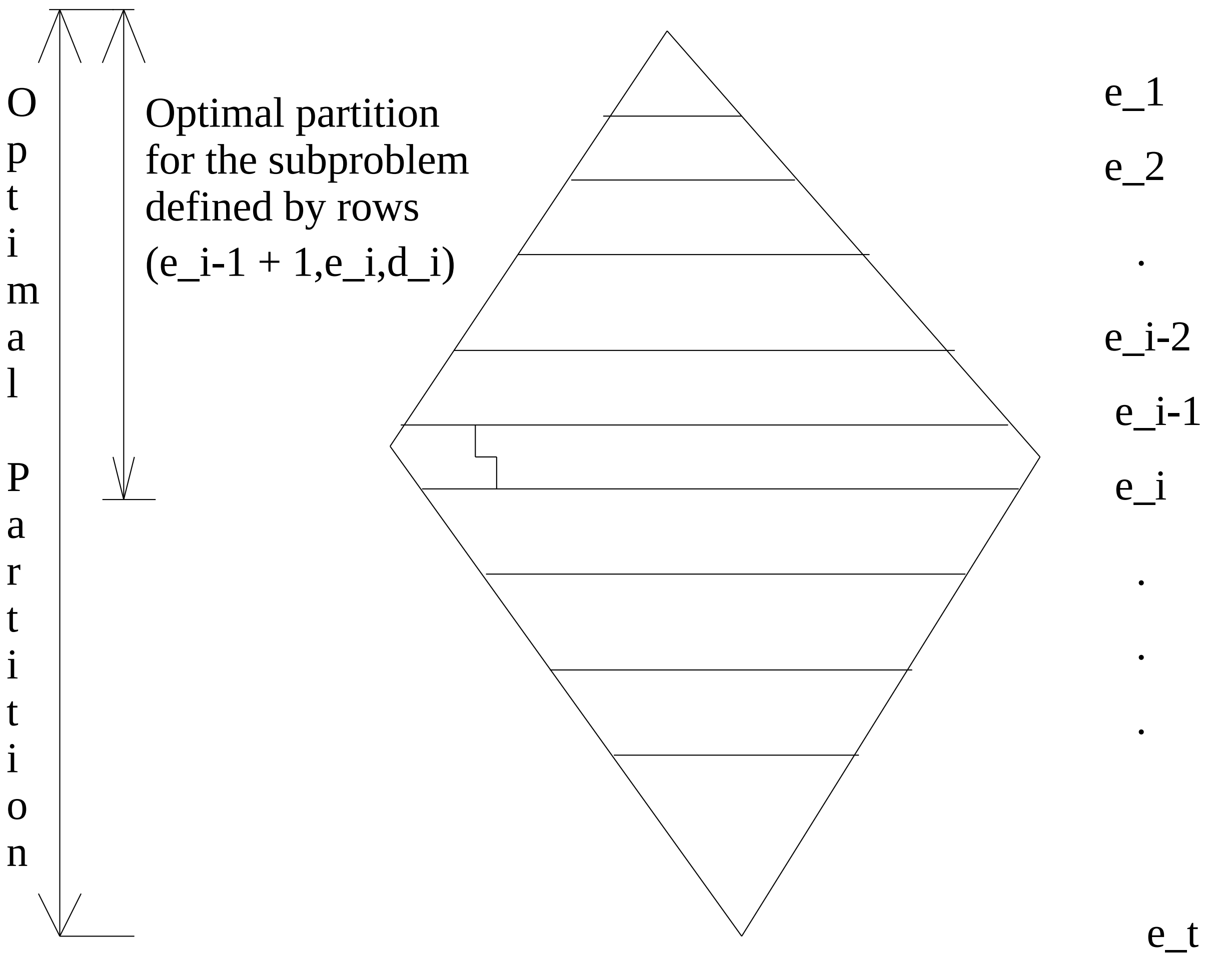

Optimal Partition
Optimal partition for the subproblem defined by rows (e_i-1 + 1,e_i,d_i)
e_1
e_2
.
e_i-2
e_i-1
e_i
.
.
.
e_t

Figure 47: Optimal Substructure

Given a grid, assume that the sequence of end-rows (e_1,e_2, ... e_{t-1}, e_t) produces an optimal solution. Now look at the subproblem ending at stripe (b_i,e_i,d_i) (see figure 47). For this subproblem, if the subsequence (e_1,e_2, ... e_{i-1}, e_i) of the original sequence didn't produce an optimal solution within the corresponding subproblem, **assuming that the last stripe was (**b_i,e_i,d_i**))**, then the optimal solution for this subproblem could be substituted into the larger solution. This new sequence would produce a better overall solution, which would contradict the fact that we previously had an optimal solution. From a graph viewpoint, the shortest path has the property that each sub-path (beginning at the the start node) is also a shortest path.

4.6 Grid Partitioning as a Shortest-Path Problem

Constructing a sequence of end-rows that partitions a grid is equivalent to determining reachability from a source to a sink by a certain path within a graph. For a given sequence of end-rows (0, r_1, r_2, ... ,r_n), we know for any consecutive end-rows, r_i, r_{i+1}, that r_{i+1} is reachable from r_i if and only if the begin-end-row pair (r_i + 1, r_{i+1}) is a valid stripe.

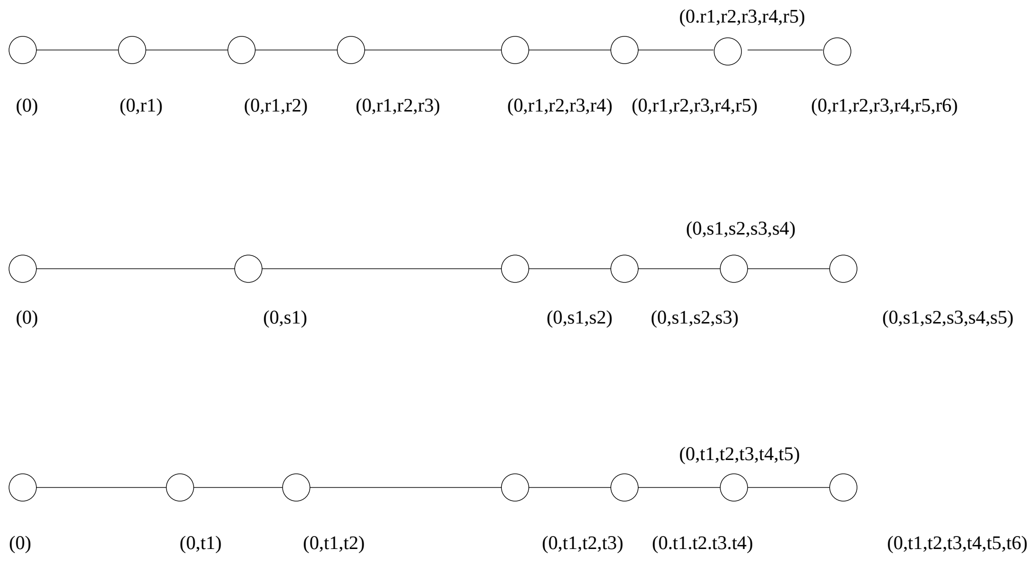
For the previous methods, the state graph for a particular grid has |V| equal to the number of valid end-row sequences for lengths 1 to M. Two vertices would have a connecting edge if and only if for vertices (0, r_1, r_2, ... ,r_i) and (0, r_1, r_2, ... ,r_i, r_{i+1})

$$(r_i + 1,\ r_{i+1}) \text{ is a valid stripe.}$$

Figure 48 contains a portion of a graph constructed using a naive approach.

In the next chapter, it will be proved that under very limited assumptions that the number of vertices in this graph is super-polynomial. What the Donaldson-Meyer (**DM**) method does is to eliminate certain redundant vertices in the state graph.

Figure 49 contains a graph for the same original grid graph, except that this graph was constructed using the DM algorithm. In figure 48, there are three paths for which the corresponding end-row sequences share final subsequences. What DM does is to collapse down those common parts of the graph. As will be shown in the next chapter, the graph for DM will have a polynomial number of vertices.



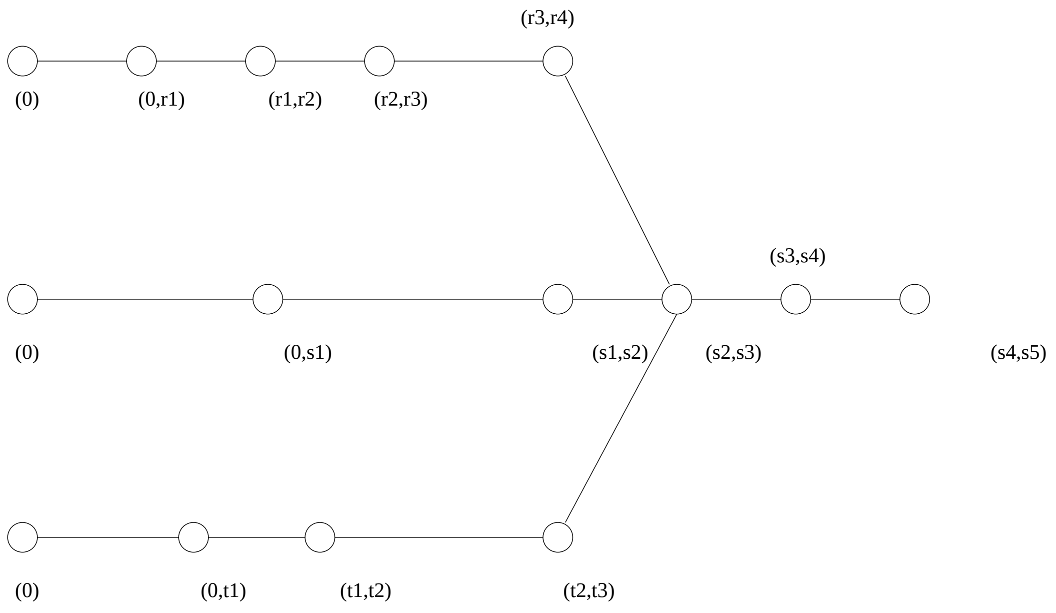
t3 = s2 = r4

t4 = s3 = r5

t5 = s4 = r6

t6 = s5 = r7

Figure 48: A naive state-graph construction.



Each node is labeled by a valid sequence of end-rows.

r4 = s2 = t3

Figure 49: The Donaldson-Meyer state-graph construction.

In the next chapter, weights are added to the graph. It will then be proved that a shortest path through this weighted graph corresponds to an optimal partition of the original grid graph.

4.7 Summary

The ability to uniquely define a subproblem by a begin-end row pair and direction of assignment will allow for the comparison of performances for multiple sequences of stripe heights. This follows from the fact that the last stripe may be assigned independently of how the subproblem to which it was appended was partitioned.

An optimal partition can be constructed by incrementally appending a stripe to a subproblem. An optimal partition possesses the qualities of common subproblems and optimal substructure.

The qualities of common subproblems and optimal substructure must be present for dynamic programming to be applicable ([CLR90], p.309). In chapter 5, both a dynamic-programming algorithm and a greedy algorithm will be presented for identifying the sequence of stripe heights that produce an optimal partition. (The methods only differ in how the data for subproblems are organized.)

The Donaldson-Meyer approach for organizing the results for subproblems eliminates redundant calculations. Figure 50 shows what happens when constructing a state graph when the property of common subproblem is not used.

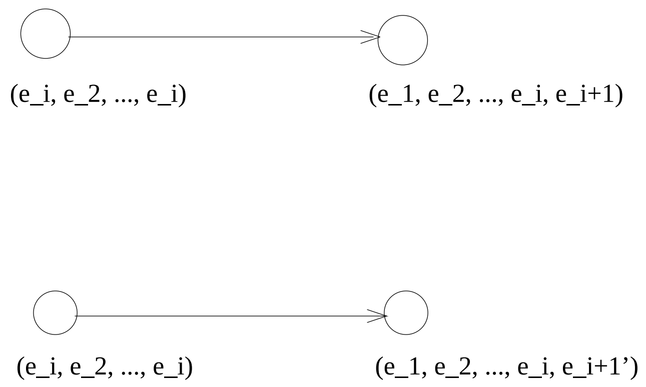


Figure 50: An example of redundant states (calculations)

To not use the common subproblem property is fatal when trying to determine the optimal sequence of stripe heights in polynomial time. In Chapter 5 it will be shown that the number of feasible stripe-height sequences is super-polynomial.

Chapter 5

Optimal Partitioning Algorithms

5.1 Introduction

From the previous section, we know that an optimal solution for a given stripe assignment possesses the properties of common subproblem and optimal substructure. Using these two properties, both a greedy algorithm and a dynamic-programming algorithm for obtaining a best stripe-based solution are presented.

The greedy algorithm is actually Dijkstra's algorithm applied to a transformed graph. It will be shown that the shortest path in this graph corresponds to an optimal solution in the original problem.

The dynamic-programming approach was the procedure used to solve the problem. The dynamic-programming recurrence models the action of stripping off the last stripe and solving a smaller, but similar, problem.

These two approaches only differ in how the solution is constructed. The greedy algorithm works bottom-up. At each step, the best new stripe is added. Eventually, all stripes containing the last row in the original grid will be considered, thus solving the problem.

The dynamic-programming algorithm works top-down, but the implementation actually is bottom-up. The dynamic-programming approach looks at all the possible final stripes and associated subproblems and selects the pair that produces an optimal solution, after recursively solving all the subproblems.

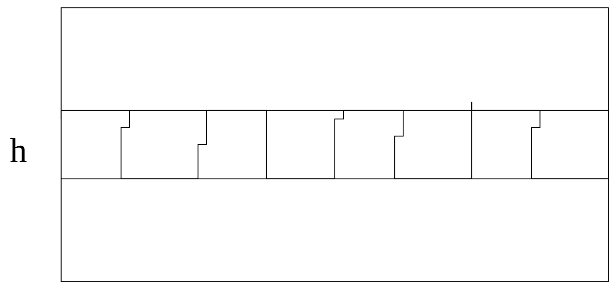
For the remainder of the chapter, it will be shown that an “brute-force” exhaustive search of feasible solutions is not practical. Then both the shortest-path and dynamic-programming approaches will be analyzed.

5.2 Stripe-Height Selection

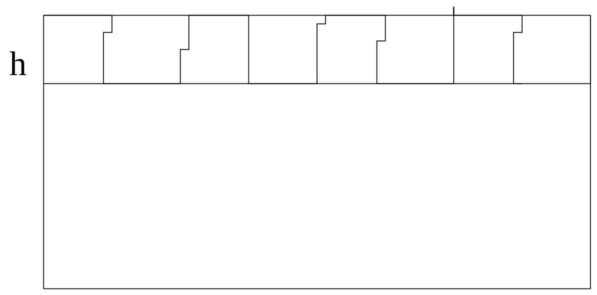
In the earlier work of Martin [Mar98] and Christou-Meyer [CM96], selection of stripe heights was the critical factor in determining the best solution possible when using a stripe-based algorithm.

Given a rectangular grid, Martin [Mar98] converts the original problem into a knapsack problem and uses knapsack software [MT90] for solving the reformulated problem. This algorithm will produce the best stripe-based solution, under the following assumptions. In addition to the rectangular grid assumption Martin [Mar98] assumes that there exists a stripe partition in which each stripe contains exactly an integral number of components. Martin [Mar98] also assumes that MN/P is an integer and that a column-wise assignment procedure is used. Under these constraints, the perimeter in each stripe is independent of the perimeter in any other stripe, and the perimeter of a stripe is independent of position within the grid.

Therefore, for a stripe of a given stripe height, there will be a unique perimeter associated with that height. The problem can then be thought of as the selection of a set of heights totaling to M with minimum total perimeter. With respect to the knapsack problem, the heights correspond to the weights and the perimeters correspond to the objective values of the associated weights. Optimal solutions of the knapsack



Sequence of components in stripe 2.



Sequence of components in stripe 1.

Figure 51: The same sequence of components appearing in different stripes.

problem correspond to optimal stripe partitions (assuming column-wise assignment and independent stripes).

For CM, the authors relax many of the restrictions (integral number of components per stripe and rectangular grids). CM doesn't in general produce optimal stripe partitions, but does generate high-quality partitions through the use of genetic-algorithm techniques.

The running time for CM can be made to be polynomial. The number of different feasible solutions that are tried can be limited to guarantee a polynomial running time.

Since Martin [Mar98] only considered rectangular domains, M, N, and P are all that are required for describing the graph. The length of the input is then O(ln M + ln N + ln P). Because Martin restricts the class of inputs to the Knapsack problems that he studied, it is not known if the resulting problem class is NP-Hard.

Whereas the type of problem that Martin [Mar98] studied could be represented with three numbers, it is assumed that a full adjacency matrix or adjacency list is required to represent a general grid graph. This assumption guarantees that that the length of input is bounded below by (V + E). Since E is O(V), we have that the length of input Ω(V) = Ω(number of cells in the grid).

That raises the question of whether or not it is possible, in a reasonable amount of time (i.e. polynomial time), to evaluate the perimeter for all the different stripe-height sequences? In this and later sections, it is assumed that the stripes will run perpendicular to the major axis of the grid. By making this assumption, bounds are then expressed in terms of the size of the original problem. Also, a result from [CM96] is needed.

The Christou-Meyer algorithm [CM96] uses the notion of **base heights**. The base height, **k**, for a given grid is equal to the floor of the square root of the area to be assigned to a processor. For this discussion, we will make the following assumptions:

1. The number of rows in the grid, M, is greater than $(k+1)^2$.
2. $k \geq 2$.
3. Stripes are perpendicular to the major axis of the grid.
4. Any set of consecutive k-1 rows contains at least one divider cell.

The first assumption is stronger than what Christou-Meyer [CM96] needed. This stronger assumption was made so that a certain inequality in the proof that follows would be easier to manipulate. Christou-Meyer [CM96] only assumed that $M \geq k(k-1)$. The second assumption is required to guarantee a valid range of stripe heights in the proof. The third assumption is needed to bound the cells in the grid by a function of M. The fourth assumption is required so that we are only counting realistic stripes. This says that there will always be enough cells within stripe *i+1* to assign any overflow from stripe *i*. This guarantees that components will only appear in at most two stripes, which is in keeping with the spirit of the striping algorithm.

If we define the number of rows in the grid to be M and the base height for each stripe to be k with the above assumptions, [CM96] showed that there exist non-negative integers α and β st.

$$M = \alpha(k) + \beta(k+1)$$

This sequence of α k's and β (k+1)'s will now be called a **base feasible solution**.

In order to discover a lower bound on the total number of valid stripe-height sequences, we need only consider sequences for which every stripe height falls within the following range:

$$[\, k-1,\ k,\ k+1,\ k+2].$$

With these assumptions, we now have the following theorem:

Theorem 5.1 *The number of feasible solutions is* Ω *(*$3^{M^{0.5}/4}$*).*

Proof

In order to prove this result, we will only look at those feasible solutions that contain S = α + β stripes. There could be feasible solutions that use a different number of stripes, but these solutions will only add to the overall count, and would only increase the lower bound.

Assume that $\alpha > \beta$ (if $\beta > \alpha$ a similar argument with β replacing α can be given). The stripes associated with α are of height k. Let's further divide the α stripes into two approximately equal groups, so that the smallest group is of size at least α/2 - 1. Call this group of stripes the **independent stripes**. The remaining α stripes along with all the β stripes will be called the **dependent stripes**.

For any stripe within the set of independent stripes, the height may increased/decreased by one or may remain the same, with respect to the base height, provided the opposite change is made to the corresponding dependent stripe. If the heights for the remaining β stripes are held constant, then we have that there are at least

$$3^{\alpha/2}/3 \geq 3^{S/4}/3 \geq 3^{M/(k+1)4}/3 \geq 3^{M^{0.5}/4}/3$$

The second inequality follows from the fact that we have:

$$\begin{aligned} \text{M} &= \alpha(\text{k}) + \beta(\text{k+1}) \\ \text{M/(k+1)} &= \alpha(\text{k})/(\text{k+1}) + \beta(\text{k+1})/(\text{k+1}) \\ &= \alpha(\text{k})/(\text{k+1}) + \beta \\ &\leq \text{S} \end{aligned}$$

The third inequality follows from the assumption that M $\geq$ (k+1)^2

□

To convert this bound into the units of the input. If M equals the height of the major axis of the rectangle that encloses the grid, we have that

$$\mathrm{M} \leq \text{(number of cells in grid)} \leq M^2$$

If the grid were a single row of cells, then M would equal the number of cells. If the grid were a square, then the number of cells equals M^2. Regardless, the number of stripes is super polynomial with respect to the number of cells in the grid.

$$3^{M^{0.5}/4}/3 \geq 3^{(no.of cells)^{0.25}/4}/3$$

From this result, we know that there are at least an super polynomial number of feasible solutions and that an exhaustive search it is not practical.

On page 71 of [Chr96] a bound on the total number of ways that the rows of a grid can be partitioned is calculated. What Christou [Chr96] counts and what is counted in theorem 5.1 of this dissertation are not the same thing. Included in Christou's result are non-feasible partitions, with respect to striping.

In [Chr96], given M rows, the 2^{M-1} figure contains stripes that may not contain enough cells to assign one processor. That means that a component could appear in three "stripes". It is not clear to the author if these are stripes that really should be considered. Theorem 5.1 discriminates as to what stripes may be considered and alters the count accordingly.

5.3 A Detailed Example

The following example will be used to demonstrate both the the shortest-path approach and the dynamic-programming approach. For the sake of constructing an example, we consider only a small number of possible stripe heights. The results presented here were generated using an implementation that differed slightly from the the algorithms to be presented. A more sophisticated approach was taken to handle U-turn regions (see Chapter 6) . Restricting the total number of stripe heights also makes presenting the shortest-path problem on one page feasible.

In chapter 4, a diamond shaped grid (see figure 39) was presented, but was not partitioned optimally. The partition in figure 52 is thought to be fairly close to the optimal partition based on striping.

The only possible improvement would come from expanding the set of valid stripe heights (Later, all valid stripes were considered. The perimeter for the partition in figure 52 is optimal, with respect to the given fill procedure. It is also easy to see that it is locally optimal. However, it is not globally optimal, since the order in the central column of 1's and 2's may be interchanged to improve the total perimeter. This illustrates the potential for even more sophisticated fill procedures.). Increasing the number of stripe heights can be done, but this introduces the possibility of violating assumptions about the number of components within a stripe. More will be said about this in Chapter 6.

In the sections to follow, it is assumed that:

1. Every stripe contains at least one divider cell.

2. For the chosen stripe assignment, the U-turn region for a given stripe is independent of the height of the previous stripe.

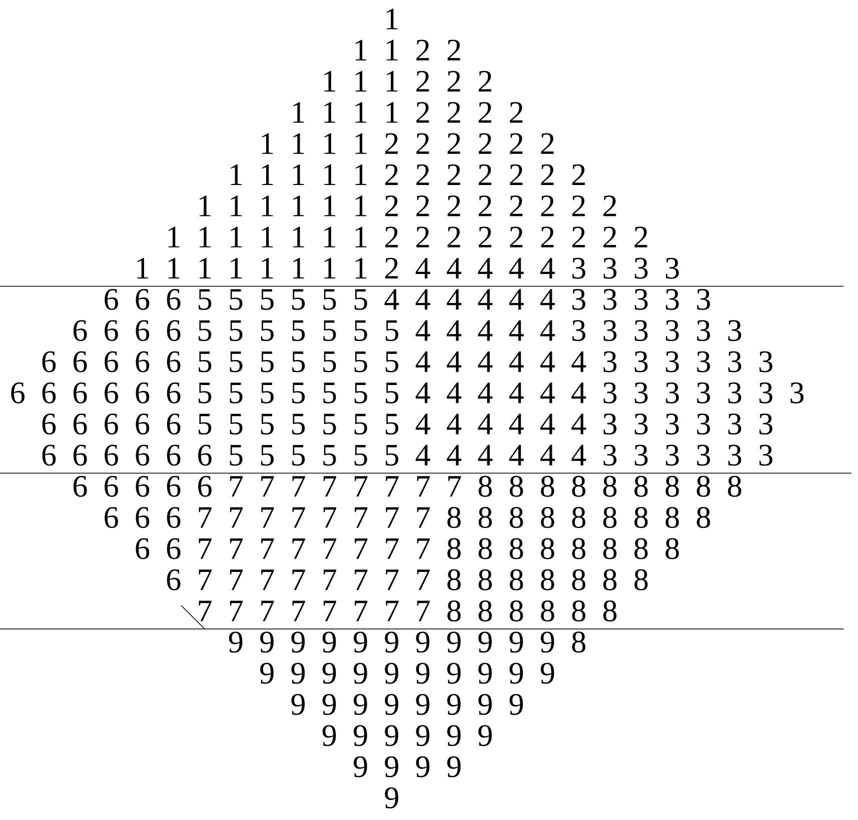


Figure 52: An optimal partition for a given set of stripe heights

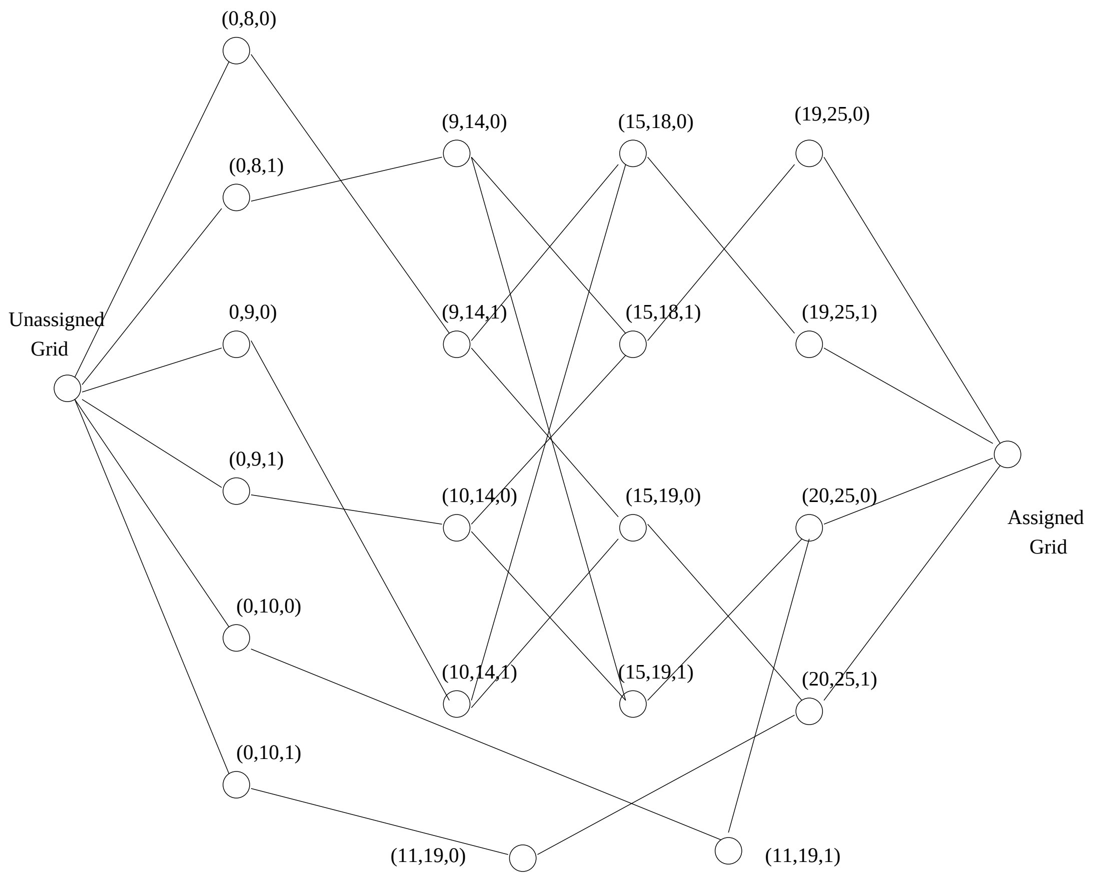


Figure 53: Partial state-graph

Subproblem Data			Previous		Subproblem	
			Stripe	Total		
b_i	e_i	d_i	Perim	Perim	b_{i-1}	e_{i-1}
0	8	0	34	34	0	0
0	8	1	36	36	0	0
0	9	0	36	36	0	0
0	9	1	36	36	0	0
0	10	0	68	68	0	0
0	10	1	68	68	0	0
9	14	0	118	154	0	8
9	14	1	120	154	0	8
10	14	0	124	160	0	9
10	14	1	124	160	0	9
11	19	0	124	192	0	10
11	19	1	124	192	0	10
15	18	0	34	188	9	14
15	18	1	34	188	9	14
15	19	0	34	188	9	14
15	19	1	34	188	9	14
19	25	0	98	286	15	18
19	25	1	98	286	15	18
20	25	0	94	282	15	19
20	25	1	94	282	15	19

Table 5: Subset of perimeter data for stripe partitioning the grid in figure 52

5.4 A State-Graph Representation of Grid Graph Partitioning

5.4.1 Background

An optimal partition of a grid graph can be thought of as a sequence of valid stripes, where each stripe is added incrementally. After each stripe is added, a new valid subproblem is generated. The discovery of an optimal solution can be thought of as going from the initial unassigned state to the final fully assigned state in an optimal fashion.

Figure 53 contains the state graph for the data in table 5.

If an optimal solution for a (sub)problem is represented by the sequence of stripe-defining end-rows (r_1, r_2, ...,r_n), the previous theorem showed that the number of states is super polynomial. To define an optimal solution, are all elements in an end-row sequence needed? No.

In chapter 4, it was shown that a begin-end-row pair and direction of assignment define a subproblem. With the sequence (r_1, r_2, ...,r_n) a lot of redundant information is carried along. The length of the sequence determines the direction of assignment. Instead of carrying along all the elements of the sequence, all that is really required is a pointer to the subproblem that the last stripe was appended to. When this is done, the size of the state graph becomes polynomial.

Figure 53 contains the state graph for the data contained in table 5 (this is the stripe information for the diamond). When the grid graph is completely unassigned, the graph is in the state corresponding to the source node, titled “unassigned grid”. The source node is connected to all nodes that correspond to a valid stripe height. Each of these nodes is connected to nodes whose corresponding subproblems are just the original problem plus a valid stripe extension. Because the graph is finite, eventually the node corresponding to the entire grid graph will be found. The path containing the cross-hatched edges corresponds to an optimal partition of the original grid graph.

For the remainder of this section, the transformation from the original grid graph to the state graph will be presented. It must be shown that for this state graph, a directed and weighted graph, the shortest path corresponds to an optimal partition of the original graph.

5.4.2 Description of a State Graph

For a **state graph**, the number of vertices equals the total number of valid subproblems. An edge connects two vertices v_i and v_{i+1} if and only if a valid stripe can be appended to the subproblem defined by v_i to get the subproblem defined by v_{i+1}.

Each node in the graph represents a unique subproblem. From the discussion on defining subproblems, three variables are required to determine a vertex. Let the ordered triple (b_i, e_i, d_{ik}) represent a given subproblem(vertex). At most, there are M different choices for b_i. Once b_i has been determined, the total number of valid stripe heights (call this number h, where h $\leq$ M, and the range of the valid stripe heights is from 1 to h) is an upper bound on the number of different possible values that e_i may assume. For any given stripe, there are only two directions that an assignment may be made. We get the following:

$$\text{total number of vertices(subproblems)} \leq 2\text{Mh}$$

The inequality comes from the fact that under certain circumstances not all M rows may be beginning rows and not all begin-end-row pairs, (b_i, b_i+k), k = 0,..,h-1, represent a valid subproblem. These situations can occur for irregularly shaped objects that contain blocks of rows that don't contain enough cells to satisfy all the assumptions. In the shortest-path problem, there will be two additional vertices: a "source" (unassigned grid) and a "sink" (assigned grid).

For a given subproblem, an upper bound on the number of different stripes that may be appended to that subproblem equals the total number of valid stripe heights, h. It follows that:

$$\begin{aligned}\text{total number of edges} &\leq \text{the total number of vertices * h}\\ &\leq 2\text{M}h^2\end{aligned}$$

For a given problem, b_i, e_i, and d_i define a subproblem. Separate this subproblem from the entire problem. Within this reduced problem, define another subproblem by b_{i-1}, e_{i-1}, and d_{i-1}. The stripe of the grid that is defined by b_{i-1}, b_i, e_i, and d_i may be assigned independently of how the subproblem defined by b_{i-1}, e_{i-1}, and d_{i-1} is assigned. Figure 54 gives an example of this construction.

The perimeter for the components assigned to the cells defined by b_{i-1}, b_i, e_i, and d_i becomes the weight of the edge connecting vertices (b_{i-1}, e_{i-1},d_{i-1}) and (b_i, e_i, d_i). Call this weight **perim(**b_{i-1}**,** b_i**,** e_i**,** d_i**)**.

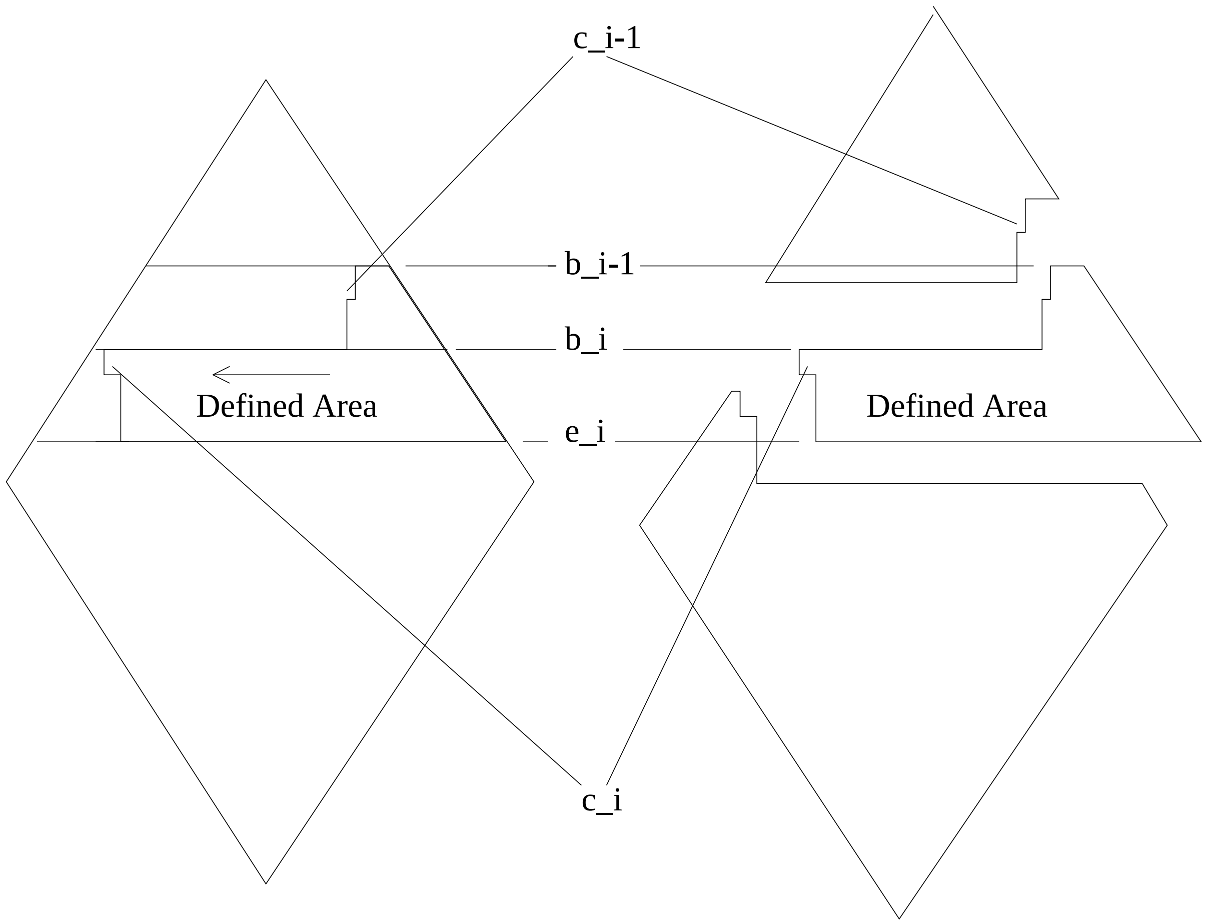


Figure 54: Area defined by b_{i-1}, b_i, e_i, and d_i

Figure 55 contains a partial example of a transformed graph. This graph will contain one source node (corresponding to an unassigned grid graph), one sink node (corresponding to an assigned grid graph) and a node for each valid subproblem. For the proofs in this dissertation, it was assumed that the initial stripe always had to be assigned left to right. For the results in table 5, the first stripe was assigned in both directions. The one-direction assumption is made to reduce the complexity of the proofs.

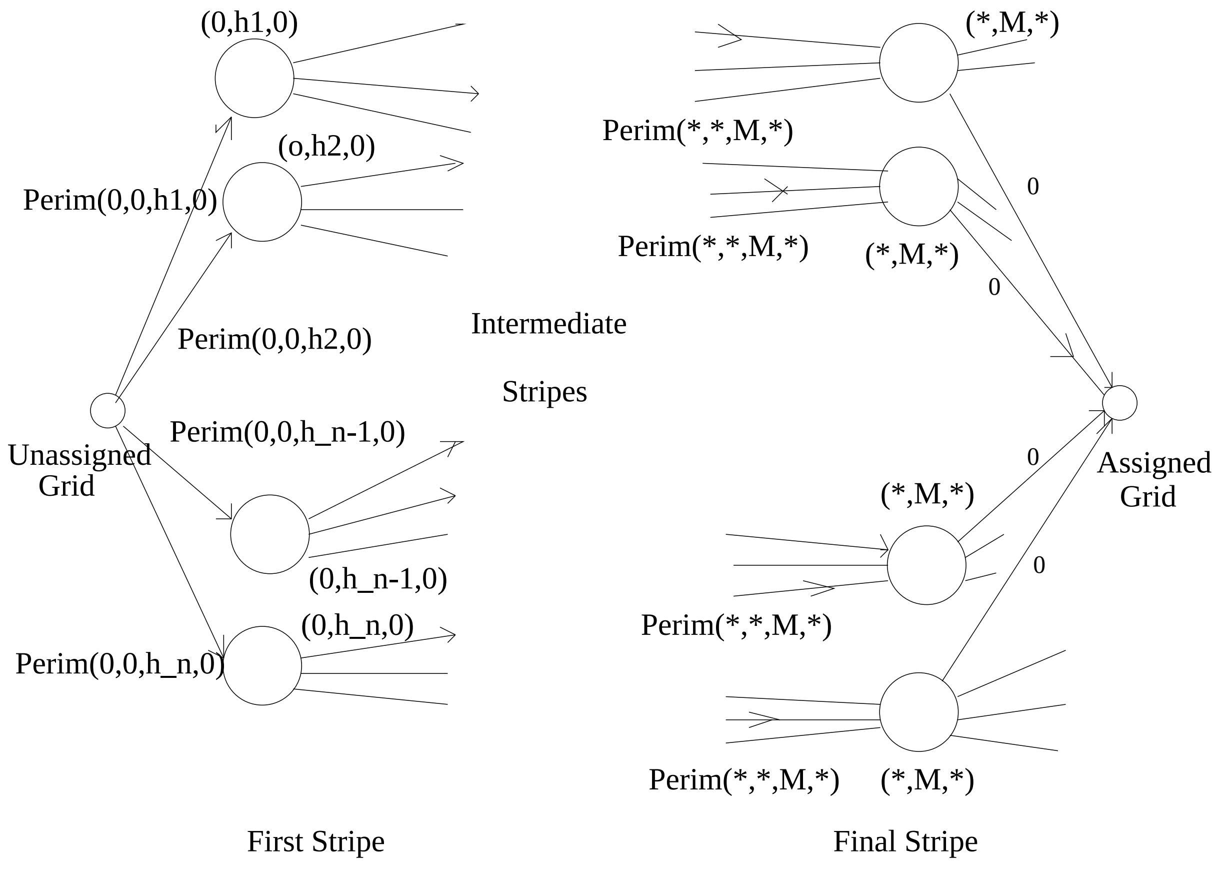


Figure 55: Transformed shortest-path graph.

5.4.3 Proof of Optimality

We must now show that the path corresponding to an optimal striping is contained within the graph.

Theorem 5.2 *The previously defined graph contains a path corresponding to every feasible sequence of end-rows. Conversely, every path beginning at node 0 corresponds to a feasible sequence of end-rows.*

Proof - (by induction)

Base Case - A sequence of length one. Given that (E) is a feasible sequence, (0,E) is a valid stripe. It follows that vertex (0,E,0) and the corresponding edge (source vertex)-(0,E,0) exists within the graph.

Induction Hypothesis - Given a sequence of end-rows of length n, there exists a path in the graph from the unassigned grid node to the node corresponding to the last stripe in the sequence.

Induction Step - Given a sequence of n+1 valid stripes, by the induction hypothesis, there exists a path from the unassigned grid node to the node corresponding to the nth stripe. Since all the n+1 stripe heights are valid, there must be an edge from the node corresponding to the nth stripe to the node corresponding to the last stripe.

The converse follows directly from the way the shortest path problem was constructed.

□

Lemma 5.1 *The shortest path in the transformed graph corresponds to an optimal stripe-based solution.*

Proof (by contradiction) - From theorem 5.2 , we know that transformed graph contains the path for an optimal solution. The value of the perimeter for the optimal striping must equal the shortest path in the graph. For if this were not the case, then the stripe heights corresponding to the shorter path would produce a better solution in the original grid graph, which would be a contradiction.

□

5.5 A Dynamic-Programming Approach

5.5.1 Background

In Chapter 4 it was shown that an optimal stripe-based partition of a grid graph possessed two qualities: optimal substructure and common subproblems. These are two properties that are present when dynamic programming is applicable. For the remainder of this this section, an optimal solution will be dissected to provide intuition as to why a recurrence can model the optimal stripe-based solution.

Table 5 contains the striping information for an optimal solution for a given small set of stripe heights and a stripe assignment process. There are four different ways that the last stripe may be assigned. The optimal solution is a minimum of these four different outcomes. An optimal solution is:

$$\begin{aligned}\text{optimal solution} &= \text{perimeter of last stripe in an optimal solution}\\ &\quad + \text{perimeter for corresponding subproblem (last stripe}\\ &\quad \text{removed)}\\ \text{Perim}(20, 25, 1) &= 94 + \text{Perim}(15,19,0)\end{aligned}$$

Notice that the value of Perim(15,19,0) must also be optimal, or else the optimal solution could be substituted, and this would be a better total perimeter than Perim(20, 25, 1), which is a contradiction.

Although the shape of the configuration corresponding to the subproblem defined by Perim(15,19,0) is no longer a diamond, the same techniques may still be applied. Again, all the different final-stripe-and-corresponding-subproblem pairs are considered, and the perimeter for the final stripe-corresponding subproblem pair that produces the best total perimeter is the value of Perim(15,19,0). It was this observation that lead to the discovery of the recurrence to be presented.

The remainder of this section will be devoted to formally defining a recurrence that models the optimal solution and to prove that the recurrence actually calculates an optimal stripe-based solution.

5.5.2 A Recurrence Relation for Grid-Graph Partitioning

We will now define a recurrence that may be used to obtain the perimeter for subproblem defined by (i,j,d), given the perimeter of smaller subproblems. This value will be stored in an array at position P(i,j,d). We need the following definitions when defining the recurrence.

Definitions

d = direction of assignment
V = number of cells in the grid
VALID-STRIPE = the set of begin-end-row pairs, (b_i, e_i), such that (e_i - b_i + 1) is less than or equal to h (the maximum allowable stripe height) and the corresponding rows contain a divider cell.
perim(prev,i,j,d) = this is the same quantity as defined in the discussion of the shortest-path material.

In the description that follows, it is assumed that all row subscripts are positive and that e_i is alway greater than or equal to b_i.

The actual recurrence is:

$$P(i,j,d) = \begin{cases} 5V & \text{if (i,j) is not in VALID-STRIPE} \\ 5V & \text{if (i = 0) and ((i,j) is in VALID-STRIPE) and (d = 1)} \\ perim(0,i,j,d) & \text{if (i = 0) and ((i,j) is in VALID-STRIPE) and (d = 0)} \\ min_b(f(b,i,j,d))+ & \\ \qquad P(b,i-1,(d+1)mod2) & \forall \text{ b s.t. (b-1,i) is in VALID-STRIPE} \end{cases}$$

where

$$f(b,i,j,d) = \begin{cases} 1 & \text{if P(b,i-1,(d+1) mod 2)} \geq 5 * \text{V} \\ perim(b,i,j,d) & \text{otherwise} \end{cases}$$

(Note: This recurrence was designed to have the first stripe to always be assigned left to right. It is easy to generalize the recurrence to allow the first stripe to be assigned in either direction.)

Now it must be shown that this alternate formulation will produce an overall perimeter equal to the optimal perimeter generated by the shortest-path approach. It may be that there is not a unique shortest path through the graph. In that case, the set of stripe heights produced by the dynamic-programming algorithm may differ from those generated using the shortest path approach.

Lemma 5.2 *For a sequence of end-rows, r_1, r_2, ..., r_n, and the corresponding array entries, P(0,r_1,0), P(r_1+1, r_2,1), ..., P(r_{n-1}+1,r_n,(n-1) mod 2)),*

$$P(r_{n-1}+1, r_n, d)) \leq Perim(r_1, r_2, \ldots r_n)$$

where Perim(r_1, r_2,...r_i) equals the perimeter of the subproblem with r_{i-1}+1, r_i as its last stripe.

Proof by Induction

(Note: In the following arguments, by default r_0 = -1. This allows the first row to begin at 0 (a holdover from programming in C)).

<u>Base Cases</u> - Sequences of length 1 and 2.

Looking at position P(0,r_1,0), the recurrence would have considered stripe (0,r_1). So we know that P(0,r_1,0) has to be less than or equal to the perimeter generated by the sequence (r_1), so P(0,r_1,0) $\leq$ Perim(r_1).

At position P(r_1+1,r_2,1), since (r_1+1, r_2) is a valid stripe, the algorithm would have considered P(0,r_1,0) + perim(0,r_1+1, r_2,1). We now have that :

$$\begin{aligned} \text{P}(r_1 + 1,r_2,1) &\leq \text{P}(0,r_1,1) + \text{perim}(0,r_1{+}1,r_2,1) \\ &\leq \text{Perim}(r_1) + \text{perim}(0,r_1{+}1,\ r_2,\ 1) \\ &= \text{Perim}(r_1,\ r_2) \end{aligned}$$

Where Perim(r_1, r_2) equals the perimeter for the sequence (r_1,r_2).

Induction Hypothesis

For a given sequence of end-rows of length n and the last two elements being r_{n-1} and r_n, the corresponding position in the array P(r_{n-1} + 1, r_n,(n-1) mod 2) will contain a perimeter no greater than the perimeter generated by the subproblem defined by the n stripes.

Inductive Step

Now assume that we have a sequence of n+1 stripes. Divide the sequence into two parts. The first part consists of the first n stripes. The second part is the last stripe. We then have the following:

$$\begin{aligned} \text{P}(r_n{+}1,r_{n+1},\text{n mod 2}) &\leq \text{P}(r_{n-1}{+}1,r_n,\text{(n-1) mod 2}) + \\ &\qquad \text{perim}(r_{n-1}{+}1,r_n{+}1,r_{n+1},\ \text{n mod 2}) \\ \text{(by Induction Hypothesis)} &\leq \text{Perim}(r_1,...,r_n) + \text{perim}(r_{n-1}{+}1,r_n,r_{n+1},\ \text{n mod 2}) \\ &= \text{Perim}(r_1,...,r_n,r_{n+1}) \end{aligned}$$

Where Perim(r_1,...,r_n+1,r_{n+1}) equals the perimeter generated by the given set of stripes.

□

Theorem 5.3 *There exists a cell in the last column of the matrix that contains the optimal perimeter using a striping fill procedure.*

Proof - For a given problem, we know that there exists a sequence of end-rows that produces an optimal filling. Define this sequence to be:

$$r_1, ..., r_{t-1}, r_t \text{ where } t \geq 1.$$

From the previous lemma, we know that P(r_{t-1}+1,r_t,(t-1) mod 2) is less than or equal to the perimeter generated by this sequence of end-rows(or end-row if t = 1). Since this sequence produces an optimal filling, we know that P(r_{t-1}+1,r_t,(t-1) mod 2) must equal the optimal filling.

□

Lemma 5.3 *For a cell in the last column that contains the minimum perimeter, the corresponding stripes generated form a feasible solution.*

Proof - by contradiction

From the previous theorem, we know that the algorithm will correctly calculate the minimum perimeter when using a striping technique. Also, the minimum perimeter must be less than or equal to 4V (this would only occur if for each cell, all of its neighbors were assigned to different processors).

Given an optimal sequence of end-rows, if the solution contained an invalid stripe height, then for some consecutive pair of end-rows, the corresponding stripe would not valid and the value of in the P matrix for that invalid subproblem would be 5V. All subsequent stripes appended to this invalid subproblem would have perimeters greater than or equal to 5V (equality would occur if additional invalid stripes were selected). Since the minimum perimeter is less than 5V, there can be no invalid stripe heights chosen.

□

Time Analysis -

The dynamic-programming array, P, is M x M, where M is the number of rows in the grid. For this structure, certain facts follow:

1. For each row of P, the number of cells that contain non-trivial stripe data is less than or equal to the number of valid stripe heights.
2. For the entire array P, it then follows that the total number of cells which will contain non-trivial stripe data is less than or equal to (number of valid stripe heights)M.

For irregularly-shaped grids, not much can be said about the number of cells within a stripe except that this number must be less than or equal to V. Each cell in the dynamic-programming array represents a unique subproblem within the grid. In order to determine the corresponding entry in the dynamic-programming array the following work must be done (This time analysis is geared for the actual implementation. The nine in the following expressions comes from the nine different assignments that are made per stripe, see Chapter 6 for more details.):

1. Creating the array of cells to be assigned when appending a stripe. There will be a different array for each allowed b_{i-1}. (Where b_{i-1} is the begin-row for the previous stripe.)

   Time = O((no. of valid stripe heights)V)

2. For each stripe-subproblem combination, make the nine different assignments.

   Time = O(9(no. of valid stripe heights)V)

3. For each stripe-subproblem and each stripe assignment, calculate the total perimeter. (This can be done by looking at each assigned cell and adding this amount to the perimeter for the subproblem.)

   Time = O(9(no. of valid stripe heights)V)

The total work for each cell is O((no. of valid stripe heights)V).

The total amount of work to fill in the non-trivial cells for the dynamic-programming array is O(M(no.of valid stripe heights)^{2}V). The amount of time to initialize the dynamic-programming array is O(M^2), which is contained within O(M(no.of valid stripe heights)^{2}V).

How does this relate to the length of input?

It was assumed that to represent this problem that an adjacency matrix or list is required. This would force the length of input to be at least equal to the number of vertices in the grid. Assuming a binary representation, that would force the length of input to be Ω(V).

This point was only mentioned because for the special case that Martin [Mar98] looked at, only three numbers are required to state the problem: the number of rows, the number of columns, and the number of components.

5.6 Summary

In polynomial time, an optimal stripe-based solution can be found, for a given stripe assignment procedure. A shortest-path approach and a dynamic-programming based method have been proven to find this solution. These discoveries are the main contribution of the research.

In previous work, the bottleneck to finding good solutions was the size of the feasible set of stripe heights. Martin was able to find an optimal stripe-based solution for rectangular grids, but the range of possible input problems was limited. Christou-Meyer produced very good results through the use of genetic algorithms. The drawback is that a solution produced Christou-Meyer was not provably optimal.

The results of this chapter combine the best of both Christou-Meyer and Martin. Both the shortest-path approach and the dynamic-programming algorithm produce provably optimal stripe-based solutions in polynomial time.

Chapter 6

Implementation and Results

6.1 Introduction

In this chapter the results of a dynamic-programming based implementation are presented. When assigning a U-turn region, nine different assignment processes are evaluated, and because of the independence of the stripe, the best stripe-assignment is kept and used in computing perimeter increment. Each of the nine stripe-assignments is a variant of the Improved U-turn algorithm (IU) that was described in Chapter 3.

For our experiments, we only consider feasible solutions such that every stripe contains a divider cell (i.e., when stripe-assigning the grid, there exist enough cells within the stripe to define a subproblem). This ensures a solution that can be modelled by a variant of the recurrence relation presented in Chapter 5.

For a rectangular grid, if a subproblem is defined by a divider cell, then the Basic U-turn algorithm can be used to produce a locally optimal solution. If the number of components within a stripe is great enough, then the Improved U-turn algorithm may be applied.

An algorithm very close to the original IU was implemented, but the results from this pilot study were not as good as expected. The idea of a U-turn region was kept, but several different stripe assignments were explored (and in the end nine assignment procedures were kept). This led to significant improvement in the computational results.

The results of Christou-Meyer (**CM**) showed that stripe-based algorithms seem to outperform other more well-known algorithms for grid graphs (see Chapter 1). To demonstrate the contributions of this research, we only need to compare against CM.

When comparing the results from this research with those of CM there are two factors at work. The first factor is the effect of expanding the set of feasible solutions that are considered (the dynamic-programming approach considers all possible stripe partitions). The second factor is the effectiveness of the nine different stripe-assignments evaluated per U-turn region.

One of the stripe-assignments is an approximation of CM's method. Because an exact CM algorithm was not implemented, CM actually performed better when the number of cells assigned to each processor was large compared to the total area of the grid. As the relative area for a component decreased, the methodology described in this dissertation (Donaldson-Meyer (**DM**)) did substantially better than CM.

6.2 Implementation

6.2.1 Subproblem Definition

In chapter 5, it was proved under certain conditions that the best stripe-height sequence could be determined. For these conditions to hold, when given a begin-end row pair and direction of assignment, a stripe assignment has to be independent of how the previous rows had been partitioned.

A stripe assignment must be able to partition the grid into two parts, each containing an integral number of components. Therefore, a stripe assignment must be able to assign an integral number of components within a stripe.

By assumption, we know that there must be a last component that can be completely assigned within the stripe (i.e. every stripe has a divider cell). Therefore, a termination point always exists within the stripe, and it is possible to ensure that the same cells within a stripe always get assigned in a certain order. We now have a means for identifying a subproblem. If the number of components within a stripe is great enough, then the Basic U-turn algorithm and the Improved U-turn algorithm can be used. Otherwise, a simple default assignment procedure can be used for the U-turn region.

At this point, we have now shown that there exists a stripe assignment that will allow for the implementation of the methodology of chapters 4 and 5.

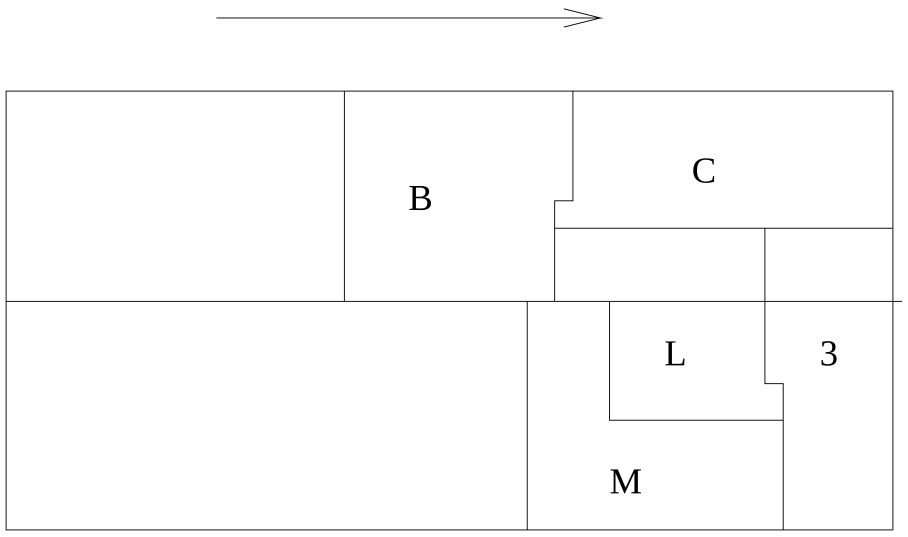


Figure 56: An Improved U-turn assignment

Theoretically, it is possible to determine an optimal stripe-height sequence that produces a locally optimal solution (either using the Improved U-turn algorithm or the Basic U-turn algorithm) for a rectangular grid (the same can't be said for non-rectangular grids). An implementation that closely resembled the Improved U-turn algorithm (except slider swaps were not made) did not produce very good results. This led to the decision to try multiple assignments per U-turn region.

6.2.2 Software and Hardware Issues

All coding was done using the C programming language, which was compiled using g++, which is a GNU project C++ compiler [ANO]. All software development was done on a Pentium Pro 200Mhz machine with 64MB of RAM. All results were generated using this Pentium machine and an Ultra Enterprise 6000 Server.

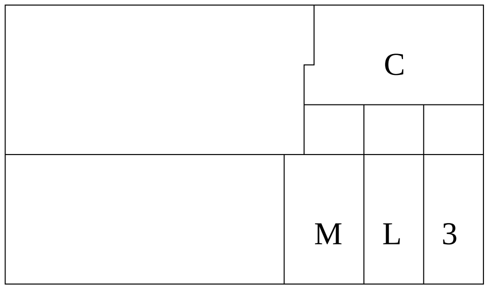
6.3 Stripe Assignments

The Improved U-turn algorithm can produce component-assignments with perimeters far from optimal. Peninsulas are still a possibility. For the stripe assignments that were examined during the course of this research, there are cases for which any single stripe assignment will do poorly. Instead of being limited by a single stripe assignment, we allowed for multiple assignments of a stripe. Figures 57 and 59 show the nine different assignment patterns that were used to assign the U-turn region. Component C is the last component that may be completely assigned to the upper stripe. In effect, by considering nine assignments, we have slightly altered the original recurrence that was presented in Chapter 5. The new recurrence is:

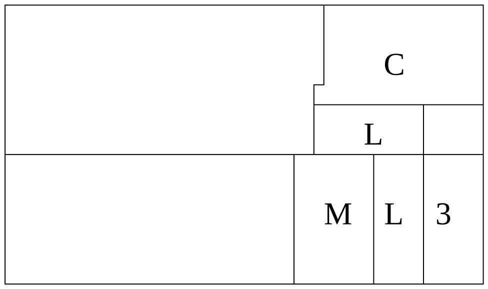
$$\text{P(i,j,d)} = min_{b,stripe\ assignment}(perim_{stripe\ assignment}\text{(b,i,j,d)} + \text{P(b,i-1,(d-1) mod 2)})$$

We have now introduced two factors for improving the quality for stripe-based solutions: choice of stripe assignment process and best stripe-height sequence.

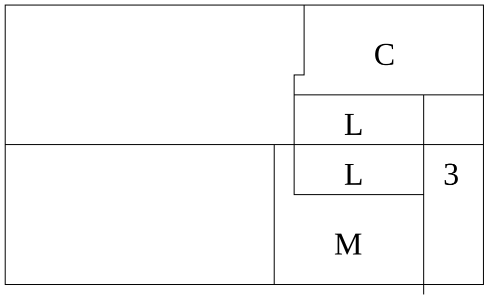
This multiple-assignment algorithm, which is modelled by the the previous recurrence, was implemented. The only fact about the stripe that we may use is that there are enough cells to complete an assignment for at least one component. The previously defined stripe assignment processes involve multiple components in the U-turn region. If



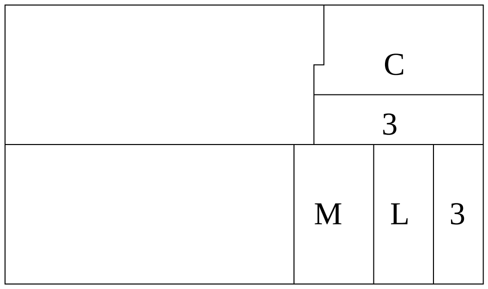
Assignment scheme 0



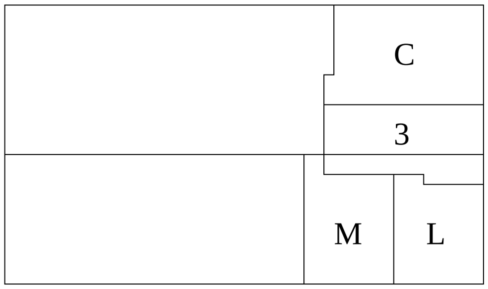
Assignment scheme 1



Assignment scheme 2



Assigment scheme 3



Assignment scheme 4

Figure 57: Stripe assignment patterns 0 - 4 (see figure 58)

For all of the assignments in this table, the C component is assigned row-wise in the upper stripe.

Number	Assignment Description
0.	All other components assigned column-wise across both stripes.
1.	All components but L and M are assigned column-wise across both stripes. In the upper stripe, L is assigned to all remaining cells and then column-wise assigned in the bottom stripe. The M component is then column-wise assigned within the bottom stripe.
2.	Same as 1 except that the L component in the bottom stripe will be assigned row-wise.
3.	3's are assigned to all remaining cells in the upper stripe. Any remaining 3's are assigned column-wise in the bottom stripe. All remaining components are assigned column-wise within the bottom stripe.
4.	3's are row-wise assigned, extending into the bottom stripe. All other components are assigned column-wise within the bottom stripe.

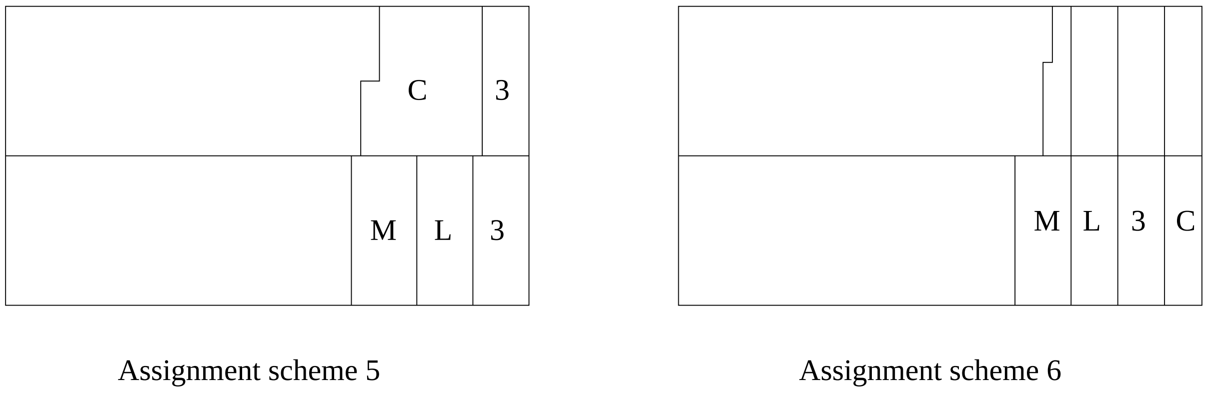
Figure 58: Striping assignments when proc C is assigned row-wise

there are not enough cells to accommodate the required number of components, then this U-turn methodology may not be applied. In this case, a default assignment was made.

At this point, the reader should understand that this modified stripe-assignment phase has no affect on the best stripe-height sequence result from the previous chapter. This is because the recurrence is suitably modified. The argument follows the along the same lines as the proof of optimality of the original shortest-path approach or dynamic-programming algorithm.

6.4 Results

The above procedure (allowing nine methods for assigning the U-turn region) was implemented. Table 6 contains the results for several non-rectangular grids comparing METIS/Pmetis/Kmetis (see Chapter 2), CM (the genetic algorithm version, again see Chapter 2), and the DM algorithm. For one of the grids partitioned into 16 components,



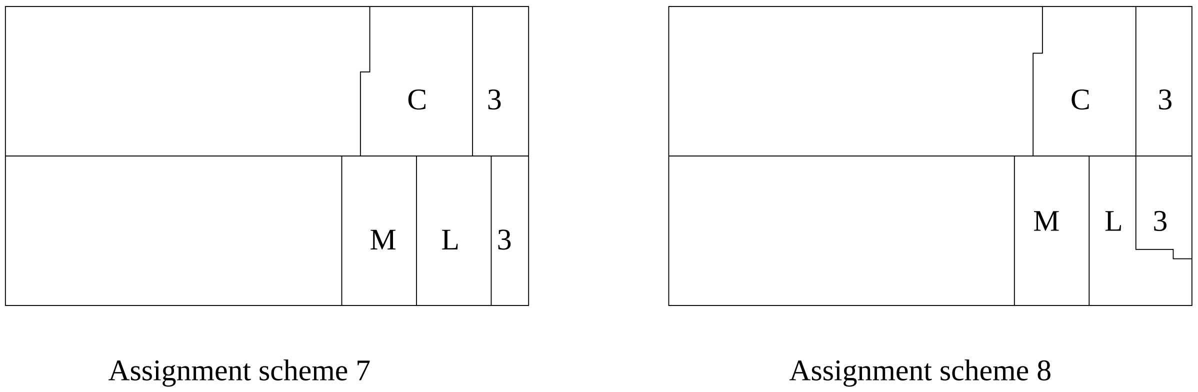


Figure 59: Stripe assignment patterns 5 - 8 (see figure 60)

Number	Assignment Description
5.	C is assigned column-wise in the upper stripe. All remaining components are assigned column-wise across both stripes.
6.	All assignments are made column-wise across both stripes in the direction that the future assignments in the bottom stripe are to be made.
7.	C is assigned column-wise in the upper stripe. The next component fills out the upper stripe, and any overflow is assigned column-wise in the bottom stripe. All remaining components are column-wise assigned in the bottom stripe.
8.	C is column-wise assigned in the upper stripe. The next component fills out the upper stripe, and any overflow is row-wise assigned in the bottom stripe. All remaining components are column-wise assigned in the bottom stripe.

Figure 60: Striping assignments when proc C is assigned column-wise

Shape	Cells	Comps	METIS	CM	DM (max stripe height, max observed height)	CM/DM Relative Improvement(%)
Diamond	4019	16	25.98	16.40	17.19 (30,20)	-4.82
Diamond	4019	64	22.07	13.37	10.25 (20,15)	23.34
Ellipse	823	16	14.58	8.33	8.33 (20,7)	0
Ellipse	823	64	5.37	5.36	3.58 (20,4)	33.21
Torus	7696	16	30.97	11.50	10.51 (40,28)	8.61
Torus	7696	64	22.59	11.00	8.38 (20,14)	23.82

Table 6: Percent above lower bound:DM, CM and METIS.

CM actually did better. There are two reasons for this.

First, CM does not require a divider cell within a stripe. That means that the size of the set of possible stripe-height sequences is greater for CM than for DM. Also, under certain conditions, CM handles overflow differently that the implementation used by DM.

When dealing with a small number of components, a poor perimeter for even one component can significantly affect the results. DM tended to have problems with the initial stripe for non-rectangular grids. As the number of components increased, the effect of a single component is reduced. To see the actual partitions produced by the DM methodology, see figures 61, 62, and 63.

Upper bounds on stripe heights were made to reduce the amount of computing. More on this is presented in the "Error Gaps" section.

As a further demonstration of the two effects, stripe-height selection and multiple assignments of the U-turn region, CMR, a non-GA version of Christou-Meyer, using the above discussed CM stripe assignment method and 100 randomly generated sets of stripe heights, was used to partition an additional set of problems. These same problems were then partitioned using the Donaldson-Meyer algorithm. Table 7 contains the results for some larger variants of the problems in Table 6. For all problems, the number of partitions was 64.

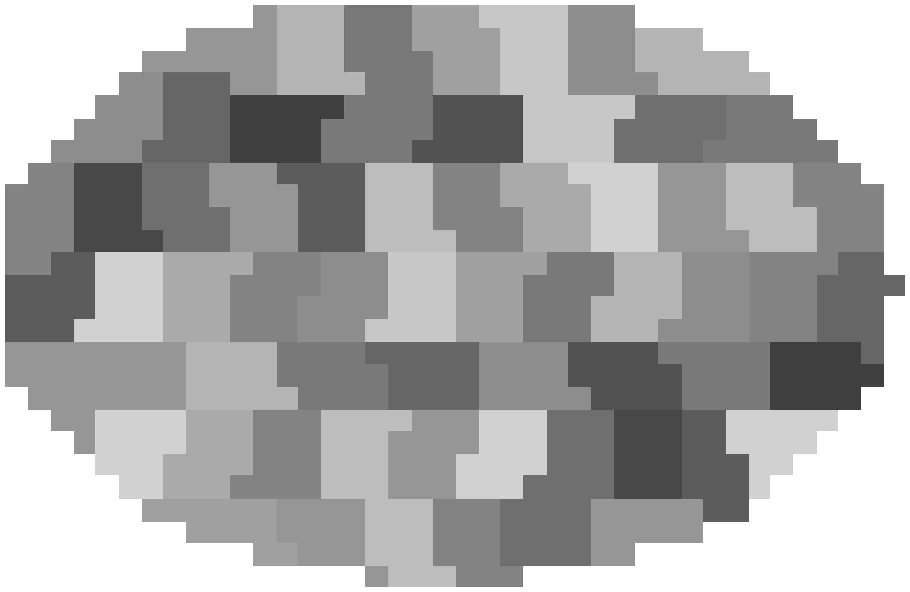


Figure 61: The DM partition of the 823-cell ellipse into 64 parts.

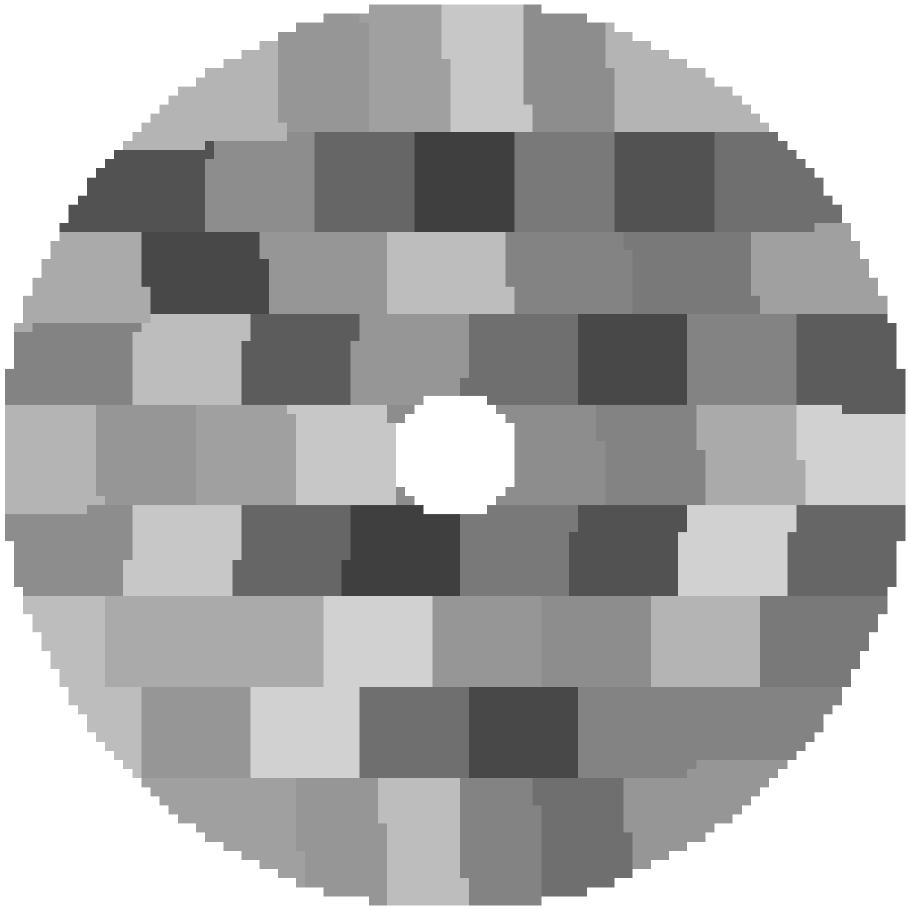


Figure 62: The DM partition of the 7696-cell torus into 64 parts.

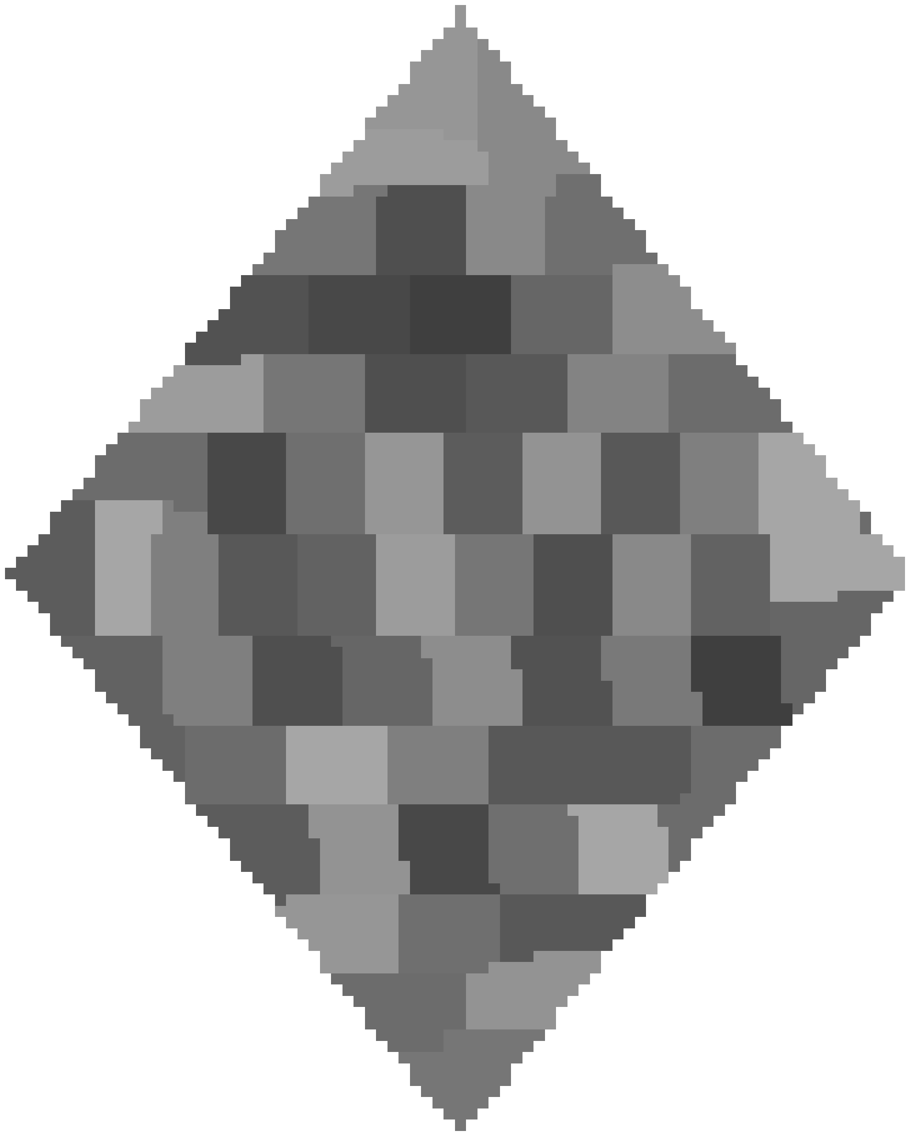


Figure 63: The DM partition of the 4019-cell diamond into 64 parts.

			Perimeters	
Shape	Cells	CM	DM (max. stripe height, max. observed height)	Lower Bound
Ellipse	1083	1198	1188(20,5)	1142
Ellipse	3305	2038	2016(20,9)	1920
Ellipse	4329	2340	2300(20,12)	2176
Diamond	3279	2112	2076(20,12)	1920
Diamond	5099	2608	2542(20,13)	2304
Diamond	12959	4100	4058(25,20)	3712
Torus	4952	2532	2468(20,13)	2304
Torus	7536	3098	3052(25,12)	2816
Torus	10980	3740	3650(25,16)	3456

Table 7: CMR versus Donaldson-Meyer

In all cases, DM outperformed CMR. However, DM took much longer than CMR for producing a result. CMR produces a solution very quickly. For a small problem (the Ellipse problem with 1083 cells) CMR took less than 8 secs, while DM took three and one-half minutes. For a big problem (the Diamond problem with 12959 cells), CM took 9 seconds; DM took about 3.7 hours.

One example that demonstrates the benefits of using the DM algorithm would be if the same inter-processor-communication intensive process were to be run many times. If the number of runs were great enough, then the additional time used by DM would be made up by reduced total run-times for the process that used the DM partition.

As was shown earlier (Chapter 2), the GA methodology of Christou-Meyer did very well when partitioning rectangular grids. As a check, the Donaldson-Meyer algorithm was applied to the 100x100 grid with 8-way partition and the 128x128 grid with 128 components. The results of Donaldson-Meyer matched those of Christou-Meyer.

Because of the complexity of the assignments to the the U-turn region, if the number of cells within a stripe is small it is not obvious if the common subproblem assumption is valid. It then follows that different assigning patterns could produce different U-turn regions.

To get around this problem, one could alter the assigning patterns to get a common U-turn region. To ensure this common substructure the following steps could be taken.

1. Apply Assignment process 0 and determine the U-turn region.

   The cells that are assigned during this phase will determine the U-turn region for the remaining eight assignments.
2. Make the other eight assignments.
3. Save the best assignment.

The process in step 1 guarantees that each assignment technique deals with exactly the same U-turn region.

6.5 Error Gaps

Since an approximation of Christou's stripe-assignment pattern is one of the nine that is tried, it is expected that CM's error gap (distance to lower bound) should in general be greater than those of DM.

To keep the running times down to reasonable levels, not all possible stripe heights were considered for many of the problems presented in this dissertation. Theoretically, if each component is to be assigned A cells, then under certain conditions, stripe heights between 1 and A (actually, stripe heights between A and M, the total number of rows, could also be considered) can be used to produce a feasible solution. But some stripe heights were thought to not produce good solutions and were not considered.

A shape for a given area A that has optimal perimeter is a square with sides of length $A^{0.5}$. The shape of the enclosing rectangle for a given area can vary and still have optimal perimeter. For the given problems the upper bounds on stripe heights were set so as to fall outside of the range of stripe heights that would produce an optimal perimeter. It is hoped that these choices for upper-bounds on stripe-heights are great enough so as to not exclude desired results. Determination of stripe-height ranges is an area of future research.

6.6 Summary

The numerical results presented in this section represent the best that can be achieved using a stripe-based algorithm, for a given set of allowable stripe heights. These results also confirm the effectiveness of the ideas of Yackel-Meyer (**YM**) and CM. CM, in particular, can produce a very good solution in a short amount of time (when considering rectangular grids Martin [Mar96] also does an outstanding job).

The techniques described here could also be applied recursively to the U-turn region for a given stripe and the associated overflow cells in the next stripe. The U-turn region and the associated overflow area form a grid graph with an integral number of components.

The run-time of the software developed for this research is dependent on problem parameters. For some of the bigger non-rectangular grids, the run-times were in the minutes (and hours for big problems). In the future (see Chapter 7, section "Stripe-height Range"), we plan to study the dependence of solution quality on the range of stripe heights used.

There are three main data structures used in the implementation. The first is an array containing the begin-end-row pairs that are valid stripes. The second is the actual grid to be partitioned (also blocks of this overall grid, corresponding to stripes, must be dynamically allocated). The third is the array used to store the dynamic-programming results. The first data structure can easily be eliminated. It is not obvious how to get around having in memory the entire grid to be partitioned, without depending on a fast file system (i.e. reading from the hard disk just those parts of the grid that are to be considered). Therefore, to work on problems of the size that Martin [Mar96] studied (e.g. rectangular 1000 x 1000 grids) would probably require a substantial amount of computing time.

Memory management is a major bottleneck in the current implementation. Using the performance measuring tool Paradyn [MCC+95] on a moderately sized problem, it appears that dynamically allocating some of the data structures may have the effect of doubling the running time. In the future, we will explore the possibility of using static, rather than dynamic, data structures.

Chapter 7

Conclusions, Contributions and Future Work

7.1 Conclusions

For a large class of problems, stripe-based methodology has proven to be a very effective approach for partitioning a grid graph. Intuitively, it is not hard to understand why these algorithms perform well. These algorithms use a greedy approach to maximize the number of components that have optimal or near-optimal perimeters. As long as the components with non-optimal perimeters are not numerous (in a relative sense), overall performance will be very good. In fact, total perimeter will approach the optimal value.

As a result of this research, we have shown that, under limited assumptions, that a rectangular grid can be assigned so as to produce a locally optimal solution.

Previous methods used a simple approach for handling those components appearing in two stripes. These methods could produce components that were far from optimal and could easily be improved by swaps, resulting in a reduction of the total perimeter for the overall assignment. This dissertation presented assignment techniques that would reduce the occurrence of these bad assignments.

In the past, it was not feasible to determine the best set of stripe heights for general domains. Under certain conditions, including rectangularity, of the domain, Martin

[Mar98] determines the best set of stripe heights, via a knapsack approach. (Because of these limitations, it is not known if the problem that Martin solved was NP-Hard.) Martin's algorithm also did not allow overflow from one stripe to the next. By using genetic algorithms, Yackel-Meyer [Yac93] and Christou-Meyer [CM96] find good sets of stripe heights, without the restrictions imposed by Martin, but their solutions do not guarantee an optimal set of stripe heights.

As a result of this research, it has been shown that there are a super polynomial number of possible sets stripe heights; and for a given stripe assignment process, a provably optimal set of stripe heights can be determined in a polynomial amount of time.

7.2 Contributions

7.2.1 Improved Stripe Assignments and Local Optimality

For the case that an integral number of components are assigned to a stripe, as a result of this research, it was shown that the assignment resulting from column-wise assignment was locally optimal. Although these assignments were not always provably globally optimal, the distance from the lower bound was guaranteed to be very small under reasonable assumptions.

If stripe heights were chosen that did not allow an integral number of components, then under the previous methodology, solutions contained assignments that could obviously be improved. The previous algorithms only allowed for a single processor to spillover across stripes. This led to the creation of peninsulas (a single row or column of cells extending from the main body of cells assigned to the component), that could be improved through a series of two-cell swaps, with the resulting assignment having a better total perimeter. This dissertation presents algorithms that do not generate such non-locally-optimal peninsulas.

The driving force behind these algorithms was the U-turn region. The U-turn region contains components that are assigned in more than one stripe and that also may have an irregular shape that forces large perimeter values. So a balance must be struck between the number of components assigned (in non U-turn regions) to near optimal configurations and the components that will be assigned (in the U-turn region) to possibly far-from-optimal configurations.

As a result of this research, nine basic assignment processes for the U-turn regions For each single process, there exist configurations of unassigned cells that will lead to poor results. However, in our experience, at least one of the nine performed relatively well in every instance.

7.2.2 Exhaustive Searches are not Feasible

Before this research, the size of the set of feasible stripe-height sequences was not established. It is now known that the size is at least super-polynomial in the number of vertices. A brute-force exhaustive search is not feasible.

7.2.3 Defining Subproblems

The driving force behind all the breakthroughs in the second half of this dissertation is the idea of defining subproblems that can be assigned independently of the remainder of the domain. Through the use of three variables and a fixed stripe assignment process, a subproblem can be uniquely defined and independently assigned. The remaining part of the original domain is also assigned independently.

This idea is actually taken one step further. Individual stripes can be identified and assigned independently of the rest of the grid. From this decomposition, two concepts have emerged. The first is the idea of a **stripe-quasi-independent (SQI)** fill procedure. For a given SQI fill procedure and a valid partial end-row sequence, the perimeter value for subproblem (e_{i-1}, e_i, d_i) equals the perimeter value for the subproblem (e_{i-2}, e_{i-1}, (d_i + 1) mod 2) plus the perimeter for the components appearing before the corresponding U-turn region for stripe i

The second concept is a **stripe-fill optimal solution (SF)**. A SF solution is optimal with respect to a set of feasible solutions obtained by:

1) specifying valid sequences of end-rows and
2) applying a SQI fill procedure.

7.2.4 Improved Stripe-Height Selection

The construction of a stripe-based partition can be thought of as incrementally appending a stripe to one subproblem and producing another subproblem. Partitions of this type posses the qualities of common subproblem and optimal substructure. These are two qualities that are present in problems for which dynamic programming and greedy algorithms are applicable. By being able to consider appropriate subproblems, both of these algorithms exploit these qualities and organize the results in such a way as to eliminate redundant and a potentially super-polynomial number of calculations.

For the greedy algorithm approach, the original problem is transformed into a state graph, with one source node (corresponding to an unassigned grid) and one sink node (corresponding to an assigned grid). Each node in the graph corresponds to a subproblem. An edge exists between node (b,e,d) and (b',e',d') if and only if a valid stripe can be appended to the subproblem defined by (b,e,d) to get (b',e',d'). The shortest path from the source to a node corresponds to an optimal partition of the corresponding subproblem. The shortest path from the source to the sink corresponds to an optimal partition in the original grid.

An optimal solution can be represented as a sequence of stripe heights and fill directions. Consider the last stripe. There are a finite number of possible final stripes. We know that the stripe height for the optimal solution must come from this set. If the final stripe corresponding to the optimal partition is stripped away, the problem remaining is a smaller version of the original problem. The partition for this smaller problem must also be optimal, or else the optimal partition could be added to the final stripe to produce a better overall solution.

This mechanism is contained in the following recurrence:

$$\text{P(i,j,d)} = min_b(\text{perim(b,i,j,d)} + \text{P(b,i-1,(d+1) mod 2)})$$

Taking into account the termination conditions and multiple stripe-assignment processes, this is the recurrence that our implementation uses.

7.3 Future Work

7.3.1 Post-Processing

As was mentioned earlier, a locally optimal solution can be improved if multi-cell (more than 2) swaps are allowed. The partition in figure 52 is locally optimal, but is not globally optimal. The key is trying to identify the expanded sets of cells to be swapped.

One method of post processing would be to use the Kernighan-Lin heuristic that allows for a sequence of swaps, in which any individual swap in the sequence may not necessarily improve the solution, but taken as a whole, the sequence of swaps does.(A worsening swap may perturb the assignments enough to create several other improving swaps) Swap-cell selections would be based on priorities determined by perimeter contributions.

Another post-processing technique to be studied will be border swapping. In figure 52, the borders for the 1's and 2's could be reassigned to get a better solution. Also, the assignment processes will be used to post-process poor quality U-turn regions and any stripes whose components are not of good quality.

7.3.2 Stripe-Height Range

The range of heights that a stripe may assume plays a major role in the amount of time required to generate an optimal set of stripe heights. As was mentioned earlier, for some of the bigger problems it took hours to generate the optimal set of stripe heights. Further study is needed on how total perimeter is affected by reductions in the set of heights that a stripe may assume.

7.3.3 Extension to the 3-D case

To extend the stripe-selection process to the 3-D grid graphs only requires a modification as to how a subproblem is to be defined. Figure 64 shows how a a 3-D graph might be divided into four "slices".

To make the diagrams a little easier to construct and to follow, only parallel pipettes will be considered. If some other type of 3-D grid were to be examined, the slicing would be done on the enclosing parallel pipette for the graph. The first two variables in any recurrence would then be the beginning and ending rows for the slice.

Within these slices, configurations approximating cubes will be packed in columns. Figure 64 shows the first column of near-cubes assigned within a stripe. This packing of cubes will continue in a manner similar to the striping done in the 2-D case. There are several different sequences of column widths that would have to be examined and organized just like in the 2-D case.

As in the 2-D case, if the last row within a 3-D subproblem is known, then the total number of complete components that may be assigned within the corresponding volume can be determined. By knowing the height of the last slice, the width and direction of assignment of the last groups of cubes packed within the slice, the configuration of cells to be assigned becomes independent of the how the previous slices were assigned.

For handling overflow from one slice to the next see figures 65 and 66

7.3.4 Parallel Computing

For all of the methodology concerning stripe selection, it was assumed that (although unstated) we had chosen the best orientation for the 2-D grid; this may not be the case. If the grid is not symmetric about both axes, then it might actually be better to use another corner as the starting point. Since there are a constant number of different starting points (four corners, when considering the enclosing frame), this type of problem lends itself to parallel computing. One way to divide up the problem is to assign each starting corner-point to a different processor. A more sophisticated variant would add a pool of post-processing tasks that could be processed in parallel. This type of distribution of work could also be extended to the 3-D case, where 24 processors could be used.

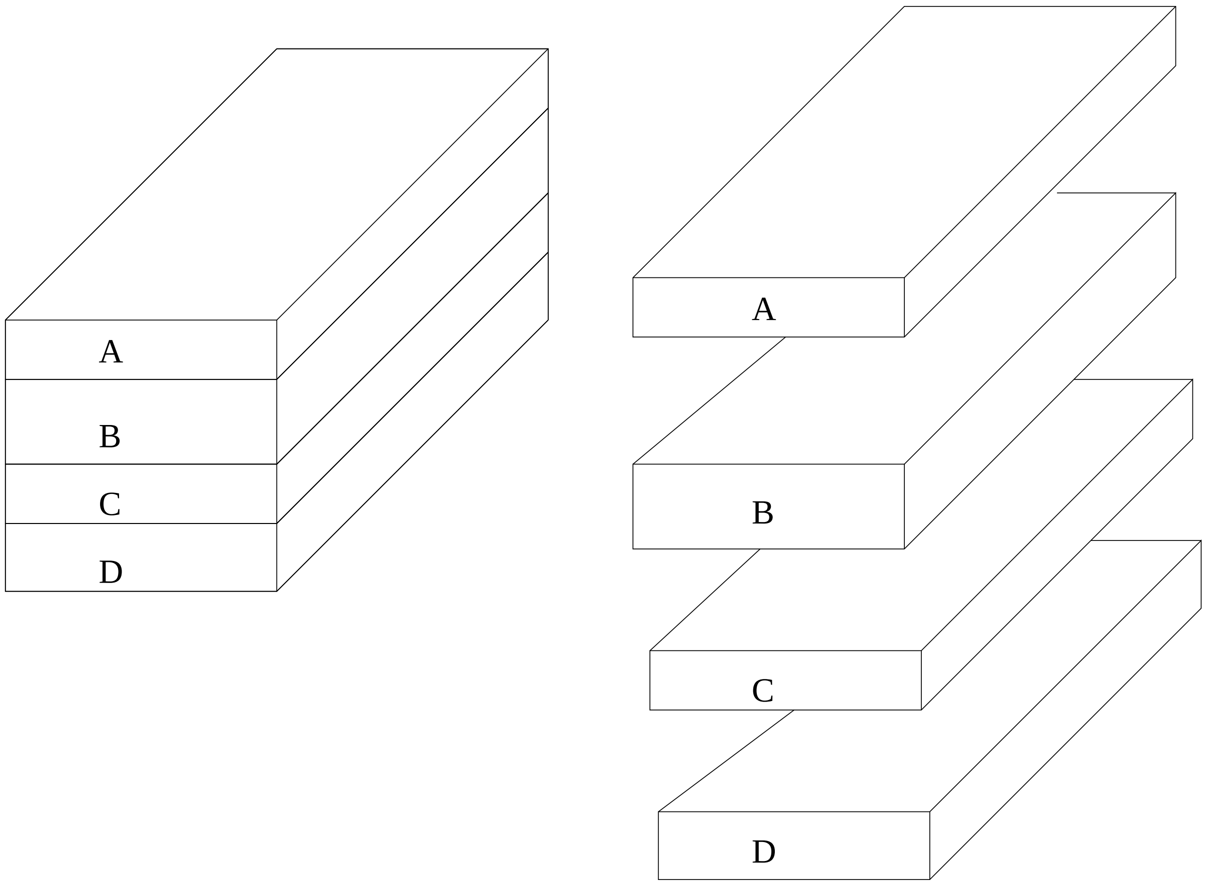


Figure 64: Slicing a 3-D grid

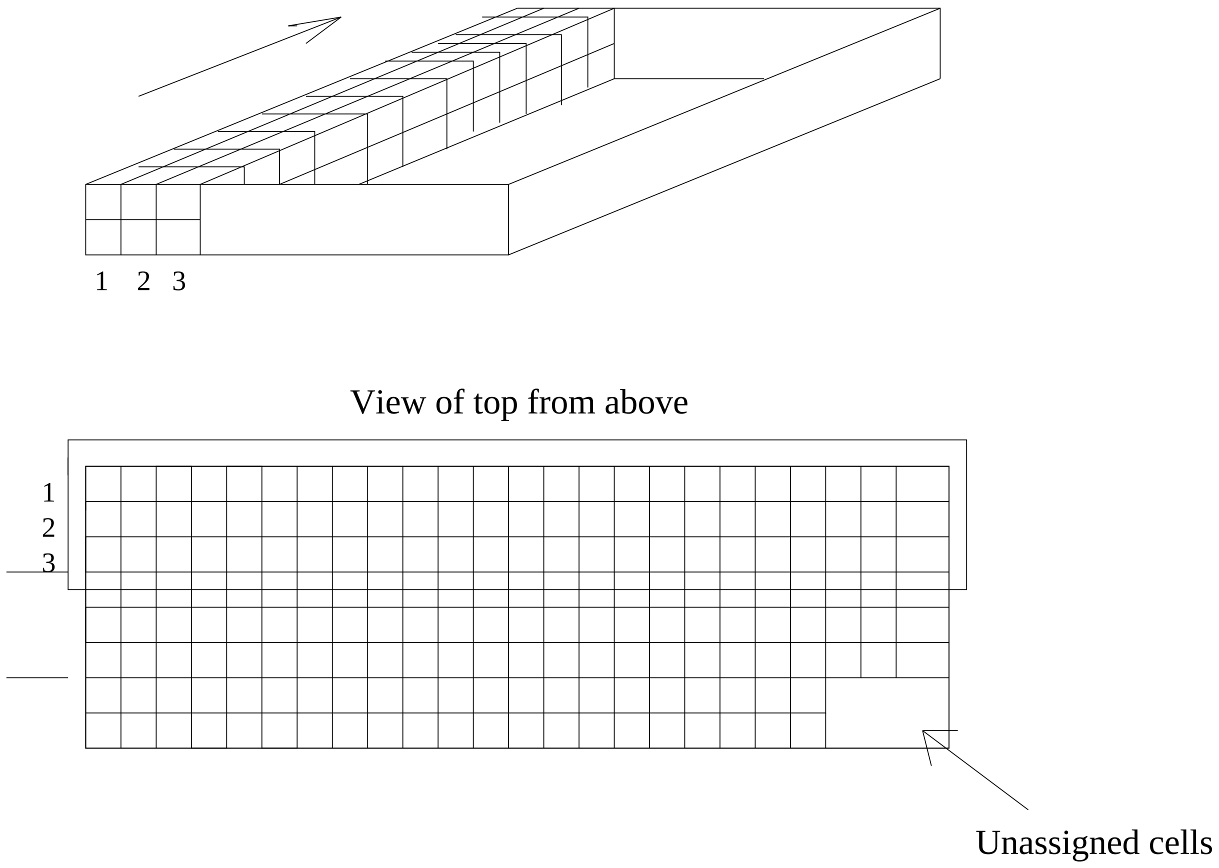


Figure 65: Assigning columns within a slice

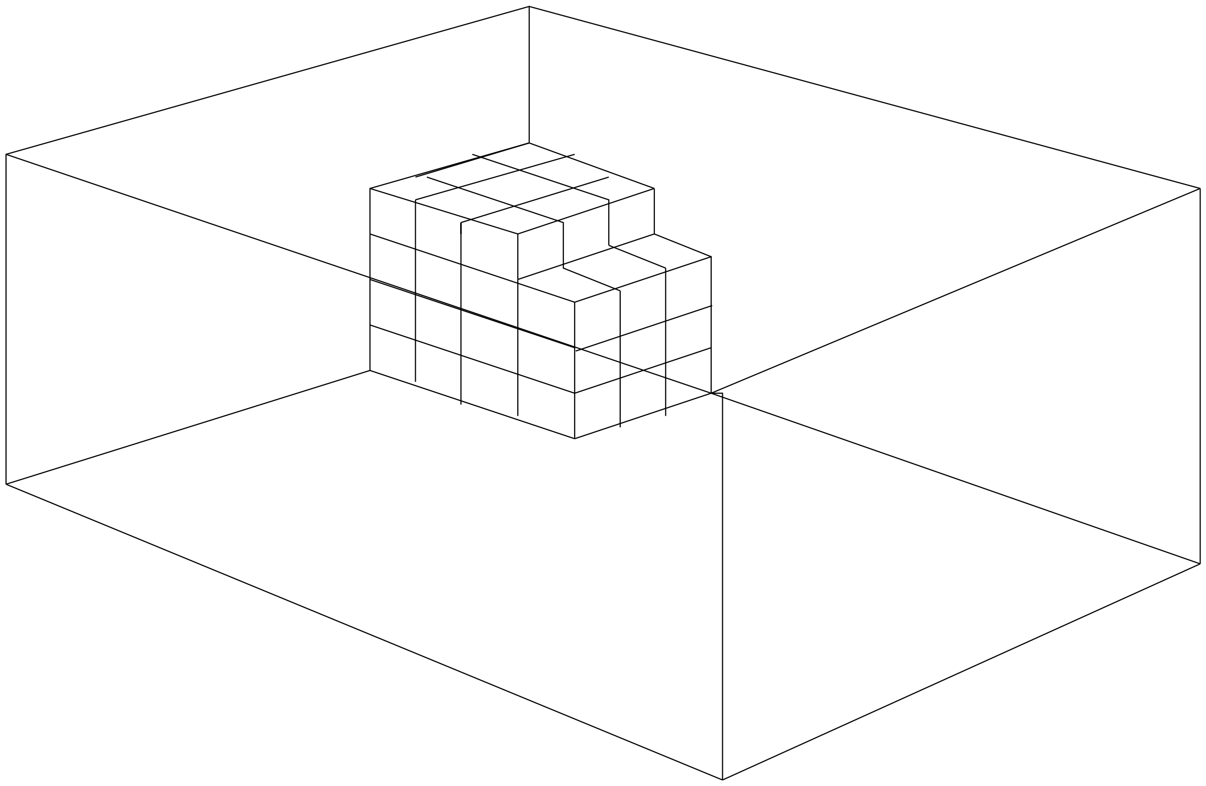


Figure 66: A configuration of unassigned cells

7.4 Summary

As a result of this research, both phases of stripe-based partitioning algorithms have been improved. The method of stripe assignment has been improved to guarantee locally optimal solutions in the rectangular case, and high-quality solutions in the general case. Previous research focussed on heuristics or restrictive approaches to select sequences of stripe heights. As a result of this research, there now exist efficient polynomial algorithms that produce the best sequence of stripe heights.

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