DEPARTMENT OF STATISTICS

University of Wisconsin
1210 W. Dayton St.
Madison, WI 53706

TECHNICAL REPORT NO. 814

July 1987

APPROXIMATE DIAGONALIZATION OF SPATIAL COVARIANCE

by

Winson Taam

and

Brian S. Yandell

Approximate Diagonalization of Spatial Covariance

by
Winson Taam and Brian S. Yandell
Department of Statistics
University of Wisconsin-Madison, U.S.A.

Abstract

We use a two-dimensional Fourier transform to diagonalize the covariance of a spatial process. Given a finite two-dimensional lattice, the Fourier matrix exactly diagonalizes the covariance when a neighborhood structure on a torus is used. When the boundary neighborhood structure is considered, the Fourier matrix approximately diagonalizes the covariance. The approximation error vanishes at a rate of 1/(width of lattice). This notion of diagonalization can then be extended to higher dimension.
Keywords: circulant matrix, Fourier matrix, Kronecker product, absolute summability, simultaneous autoregressive, conditional autoregressive.

1. Introduction

The use of Fourier matrices to diagonalize the general covariance matrix of a time series has been documented in Fuller (1976), but a similar result has not been obtained for spatial series. The idea of two dimensional spectral analysis was considered in Cliff and Ord (1981), Ripley (1981), Priestley (1984), and others. This Fourier transform (covariance diagonalization) of lattice data yields an independent set of periodograms. The Fourier matrix which performs the transformation comes from a eigenvalue-eigenvector decomposition of a circulant matrix. In practice, the assumption of a circular lattice (torus) is sometimes questionable despite the fact that a torus lattice has nice Markovian properties, (cf. Moran, 1973a,b). Our goal here is to establish the result that the discrete Fourier transform on a circular (torus) lattice provides a close approximation to the one on a non-cirular (non-torus) lattice.

The motivation of this work comes from three sources. First, the diagonalization of the covariance of a simultaneous autoregressive process on a finite torus raises the question of how to diagonalize a covariance not from a torus neighborhood structure. Secondly, the Fourier transform on a non-circular lattice raises also the question of independence among the periodograms. The answer to these two problems for time series can be found in Fuller (1976, chapter 4). The last question we wish to address is how to extend this idea to higher dimensions.

This paper uses circulant matrices and Kronecker products as the building blocks to answer the above questions. In the next section, the basic definitions and notations are laid out. In section four, the two-dimensional neighborhood matrices are defined. The third and the fifth sections study the diagonalization in one- and two-dimensional lattices, respectively. In section six, extension to higher dimensions is considered. The last section discusses some examples and applications of this work.

2. Notation for the Building Blocks

Let us define the following $n \times n$ matrices:

$$\mathbf{B}_n = \begin{bmatrix} 0 & \mathbf{I}_{n-1} \\ 0 & 0 \end{bmatrix} \tag{2.1}$$

$$\Pi_n = \begin{bmatrix} 0 & \mathbf{I}_{n-1} \\ 1 & 0 \end{bmatrix} = \mathbf{B}_n + \mathbf{B}_n^{(n-1)\mathrm{T}} \ . \tag{2.2}$$

Note that Π_n is a circulant matrix as defined in Davis (1979). It has the following eigenvalue-eigenvector decomposition:

$$\Pi_n = \mathbf{P}\Lambda_n\mathbf{P}^* \quad \text{and} \quad \Pi_n{}^{\mathrm{T}} = \mathbf{P}\Lambda_n{}^*\mathbf{P}^* \tag{2.3}$$

where Λ_n is diagonal and

$$\{\mathbf{P}\}_{jk} = \frac{1}{\sqrt{n}}\exp(i\frac{2\pi}{n}jk) \ , \quad \{\Lambda_n\}_{kk} = \exp(i\frac{2\pi}{n}k) \ , \quad \{\Lambda_n{}^*\}_{kk} = \exp(-i\frac{2\pi}{n}k) \ ,$$

$j,k=0,1,2,\cdots n-1$ (* denote conjugate transpose). Let us introduce the matrix functionals $\mathbf{J}$ and $\mathbf{F}$,

$$\mathbf{J}(x,n) = \begin{cases} \mathbf{I}_n & \text{if } x = 0 \\ \mathbf{B}_n^x & \text{if } x > 0 \\ \mathbf{B}_n^{-x\mathrm{T}} & \text{if } x < 0 \end{cases} \tag{2.4}$$

and

$$\mathbf{F}(x,n) = \begin{cases} \mathbf{I}_n & \text{if } x = 0 \\ \Pi_n^x & \text{if } x > 0 \ . \\ \Pi_n^{-x\mathrm{T}} & \text{if } x < 0 \end{cases} \tag{2.5}$$

Note some of the properties and relationships of these matrices:

$$\mathbf{B}_n^n = 0 \ , \ \Pi_n^n = \mathbf{I}_n \ , \text{ and} \tag{2.6}$$

$$\Pi_n^j = \begin{bmatrix} 0 & \mathbf{I}_{n-j} \\ \mathbf{I}_j & 0 \end{bmatrix} = \mathbf{B}_n^j + \mathbf{B}_n^{(n-j)\mathrm{T}} \text{ for } j \le n \ .$$

Therefore, $\mathbf{F}$ in (2.5) can be written as

$$\mathbf{F}(x,n) = \begin{cases} \mathbf{J}(0,n) & \text{if } x = 0 \\ \mathbf{J}(x,n) + \mathbf{J}(x-n,n) & \text{if } x > 0 \\ \mathbf{J}(x,n) + \mathbf{J}(n-x,n) & \text{if } x < 0 \end{cases} \tag{2.7}$$

3. One Dimensional Process

Suppose that we have a sample of n observations from a stationary time series $\{ Y_t \}$. We can express the covariance matrix Γ of these n observations in terms of (2.1),

$$\Gamma = \gamma(0)\mathbf{I}_N + \sum_{j=1}^{n-1}\gamma(j)(\mathbf{B}_n^j + \mathbf{B}_n^{j\mathrm{T}}) \ , \quad \textit{where} \quad \gamma(j) = Cov(Y_t, Y_{t-j}) = Cov(Y_t, Y_{t+j}) \ . \tag{3.1}$$

To approximately diagonalize this covariance, one uses the Fourier matrix $\mathbf{P}$ in (2.3). In fact, the Fourier matrix diagonalizes Π_n in (2.2) exactly. That is, $\mathbf{P}$ diagonalizes the circulant counterpart of Γ. Let us define this circular covariance matrix Γ_s as

$$\Gamma_s = \gamma(0)\mathbf{I}_N + \sum_{j=1}^{m}\gamma(j)(\Pi_n^j + \Pi_n^{j\mathrm{T}}) \tag{3.2}$$

where $\gamma(.)$ is as in (3.1) and $m=[n/2]$, with $[x]$ being the integer part of x.

Fuller (1976) showed in Theorem 4.2.1 that with stationarity on Y_t and absolute summability on $\gamma(.)$,

$$\| \mathbf{P}^*(\Gamma_s - \Gamma)\mathbf{P} \| \to 0 \ \ as \ \ n \to \infty \tag{3.3}$$

for $\|.\|$ being the matrix norm and $\mathbf{P}$, Γ and Γ_s as defined in (2.3), (3.1) and (3.2). We will sketch the proof by using the building blocks defined in the previous section.

To see that $\mathbf{P}$ approximately diagonalizes Γ, let us assume n is odd. Then

$$\Gamma_s - \Gamma = \sum_{j=1}^{m} [\gamma(j) - \gamma(n-j)](\mathbf{B}_n^{(n-j)} + \mathbf{B}_n^{(n-j)T}) \tag{3.4}$$

and

$$\{\mathbf{P}^* \mathbf{B}_n^{(n-j)} \mathbf{P}\}_{lk} = \begin{cases} \dfrac{1}{n} \dfrac{1-exp\,(i\frac{2\pi}{n}(k-l)j)}{1-exp\,(i\frac{2\pi}{n}(k-l))}\, exp\,(-i\frac{2\pi}{n}j) & \text{if } k \neq l \\ \dfrac{j}{n} exp\,(-i\frac{2\pi}{n}j) & \text{if } k=l \end{cases}$$

Let $\mathbf{P}_{.l}^*$ denote the l th row of $\mathbf{P}^*$ and let $\mathbf{P}_{.k}$ denote the k th column of $\mathbf{P}$. Since

$$|\mathbf{P}_{.l}^*(\Gamma_s-\Gamma)\mathbf{P}_{.k}| \le |\mathbf{P}_{.k}^*(\Gamma_s-\Gamma)\mathbf{P}_{.k}| \text{ for all } l, \tag{3.5}$$

it suffices to show that

$$|\mathbf{P}_{.k}^*(\Gamma_s-\Gamma)\mathbf{P}_{.k}| \to 0 \;\; as \;\; n \to \infty \text{ for all } k\,.$$

The term on the right of (3.5) can be written as

$$|\sum_{j=1}^{m}[\gamma(j)-\gamma(n-j)]\frac{2j}{n}cos(\frac{2\pi}{n}jk)| \le \sum_{j=1}^{m}|\gamma(j)-\gamma(n-j)|\frac{2j}{n} \tag{3.6}$$

$$\le \sum_{j=1}^{m}\frac{2j}{n}|\gamma(j)| + \sum_{j=n-m}^{n-1}\frac{2m}{n}|\gamma(j)|$$

and the first term on the right goes to zero because of the Kronecker Lemma (see Fuller (1976, p.138) for details of this lemma). The second term goes to zero because of the absolute summability of $\gamma(.)$.

When n is even, replace m in the equation (3.4) by $m-1$, and the convergence to zero in (3.6) still holds. Therefore, the $\mathbf{P}$ matrix approximately diagonalizes the covariance Γ in (3.1) if the sample is large.

Remark. The one dimensional bilateral scheme (two directional correlation) can be represented by a one dimensional unilateral scheme (Whittle, 1954). Hence, the bilateral symmetric scheme in one-dimension has this approximate diagonalization of the covariance. The only change required for an asymmetric process would be the $\gamma(.)$ function. For example, one can define a bilateral scheme with different covariances in different directions as:

$$\Gamma = \gamma(0) + \sum_{j=1}^{n-1}\gamma_1(j)\mathbf{B}_n^j + \gamma_2(j)\mathbf{B}_n^{jT},$$

where $\gamma(0) = Var\,(Y_t)$, $\gamma_1(j) = Cov\,(Y_t, Y_{t+j})$ and $\gamma_2(j) = Cov\,(Y_t, Y_{t-j})$. The same result of (3.3) can be shown easily because the proof is based on the matrix structure, and this Γ essentially has the same structure as before. What is needed here is the absolute summability on both γ_1 and γ_2.

4. Two Dimensional Neighborhood Structure

Before we define the two-dimensional neighborhood matrices, let us establish a modular indexing system. Suppose that we have an $r \times c$ two-dimensional lattice. Let $N = rc$. We string the lattice out by rows. That is,

$$j = u_j c + v_j \;\text{ with }\; u_j = 0,1,\cdots r-1\;,\;\; v_j = 0,1,\cdots c-1\;,\text{ and }\; j = 0,1,2\cdots(r-1)(c-1). \tag{4.1}$$

We will use this indexing scheme throughout this paper. The following matrices are therefore $N \times N$. Let us define the circular (torus) neighborhood matrices

$$\mathbf{W}_1^{(j)} = \mathbf{I}_r \otimes (\Pi_c^j + \Pi_c^{j\mathrm{T}}) = \mathbf{F}(0,r) \otimes (\mathbf{F}(j,c) + \mathbf{F}(-j,c)) \ , \tag{4.2}$$

$$\mathbf{W}_2^{(j)} = (\Pi_r^j + \Pi_r^{j\mathrm{T}}) \otimes \mathbf{I}_c = (\mathbf{F}(j,r) + \mathbf{F}(-j,r)) \otimes \mathbf{F}(0,c) \ ,$$

$$\mathbf{W}_3^{(j,k)} = (\Pi_r^j \otimes \Pi_c^k) + (\Pi_r^{j\mathrm{T}} \otimes \Pi_c^{k\mathrm{T}}) = (\mathbf{F}(j,r) \otimes \mathbf{F}(k,c)) + (\mathbf{F}(-j,r) \otimes \mathbf{F}(-k,c)) \ ,$$

and

$$\mathbf{W}_4^{(j,k)} = (\Pi_r^j \otimes \Pi_c^{k\mathrm{T}}) + (\Pi_r^{j\mathrm{T}} \otimes \Pi_c^k) = (\mathbf{F}(j,r) \otimes \mathbf{F}(-k,c)) + (\mathbf{F}(-j,r) \otimes \mathbf{F}(k,c)) \ .$$

$\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4$ represent the horizontal, vertical, and the two diagonal neighborhoods, respectively. Figure A illustrates the idea. Let us use **M** to denote the non-circular neighborhood matrices:

$$\mathbf{M}_1^{(j)} = \mathbf{I}_r \otimes (\mathbf{B}_c^j + \mathbf{B}_c^{j\mathrm{T}}) \ , \ \mathbf{M}_2^{(j)} = (\mathbf{B}_r^j + \mathbf{B}_r^{j\mathrm{T}}) \otimes \mathbf{I}_c \ , \tag{4.3}$$

$$\mathbf{M}_3^{(j,k)} = (\mathbf{B}_r^j \otimes \mathbf{B}_c^k) + (\mathbf{B}_r^{j\mathrm{T}} \otimes \mathbf{B}_c^{k\mathrm{T}}) \ , \text{ and } \mathbf{M}_4^{(j,k)} = (\mathbf{B}_r^j \otimes \mathbf{B}_c^{k\mathrm{T}}) + (\mathbf{B}_r^{j\mathrm{T}} \otimes \mathbf{B}_c^k) \ .$$

Note that the **J** notations in (2.4) can be used as well.

The orthogonal matrix $\mathbf{P}_N$ that diagonalizes the **W**'s is the Kronecker product of the matrices that diagonalize the Π's. That is,

$$\mathbf{P}_N = \mathbf{P}_r \otimes \mathbf{P}_c \quad \text{and} \quad \{\mathbf{P}_N\}_{jk} = \frac{1}{\sqrt{rc}} \exp(i(\frac{2\pi}{r} u_j u_k + \frac{2\pi}{c} v_j v_k)), \tag{4.4}$$

where u_j, u_k, v_j, v_k are the indices of (4.1). Hence,

$$\mathbf{P}_N (\Pi_r^j \otimes \Pi_c^k) \mathbf{P}_N^* = (\mathbf{P}_r \otimes \mathbf{P}_c)(\Pi_r^j \otimes \Pi_c^k)(\mathbf{P}_r^* \otimes \mathbf{P}_c^*)$$

$$= (\mathbf{P}_r \Pi_r^j \mathbf{P}_r^*) \otimes (\mathbf{P}_c \Pi_c^k \mathbf{P}_c^*)$$

$$= \Lambda_r^j \otimes \Lambda_c^k \ ,$$

where $j \le m_r$ and $k \le m_c$ with $m_r = [r/2]$ and $m_c = [c/2]$. Λ_r and Λ_c are Λ_n defined in (2.3) with n replaced by r and c, respectively.

One of the most common nearest-neighborhood matrices for a torus is the root case, i.e.

$$\mathbf{W} = \mathbf{I}_r \otimes (\Pi_c + \Pi_c^{\mathrm{T}}) + (\Pi_r + \Pi_r^{\mathrm{T}}) \otimes \mathbf{I}_c = \mathbf{W}_1^{(1)} + \mathbf{W}_2^{(1)} \ ,$$

which is diagonalized by $\mathbf{P}_N$,

$$\mathbf{P}_N \mathbf{W} \mathbf{P}_N^* = \Lambda_N = \mathbf{I}_r \otimes (\Lambda_c + \Lambda_c^*) + (\Lambda_r + \Lambda_r^*) \otimes \mathbf{I}_c$$

with

$$\{\Lambda_N\}_{kk} = 2[\cos(\frac{2\pi}{r} u_k) + \cos(\frac{2\pi}{c} v_k)] \ .$$

There is an exact diagonalization of the non-circular counterpart of **W**. Let us denote the non-circular first step neigborhood matrix as **M**,

$$\mathbf{M} = \mathbf{I}_r \otimes (\mathbf{B}_c + \mathbf{B}_c^{\mathrm{T}}) + (\mathbf{B}_r + \mathbf{B}_r^{\mathrm{T}}) \otimes \mathbf{I}_c = \mathbf{M}_1^{(1)} + \mathbf{M}_2^{(1)} \ .$$

This is the one step rook case neighborhood matrix with unbalanced weights on the boundary of the lattice. Since there is an exact diagonalization of $\mathbf{B}_n + \mathbf{B}_n^{\mathrm{T}}$ (Conte and deBoor, 1980, p.206),

$$\mathbf{Q}_n \Psi_n \mathbf{Q}_n^{\mathrm{T}} = (\mathbf{B}_n + \mathbf{B}_n^{\mathrm{T}})$$

where the orthogonal matrix $\mathbf{Q}_n$ and the diagonal matrix Ψ_n are defined as

$$\{\mathbf{Q}_n\}_{jk} = (\frac{n+1}{2})^{-1/2} \sin(\frac{\pi}{(n+1)} jk) \quad \text{and} \quad \{\Psi_n\}_{kk} = 2\cos(\frac{\pi}{(n+1)} k) \quad j,k = 1,2 \cdots n \ ,$$

the exact diagonalization of $\mathbf{M}$ arises from

$$\mathbf{M} = \mathbf{I}_r \otimes (\mathbf{Q}_c \Psi_c \mathbf{Q}_c^{\mathrm{T}}) + (\mathbf{Q}_r \Psi_r \mathbf{Q}_r^{\mathrm{T}}) \otimes \mathbf{I}_c$$

$$= (\mathbf{Q}_r \otimes \mathbf{Q}_c)(\mathbf{I}_r \otimes \Psi_c)(\mathbf{Q}_r \otimes \mathbf{Q}_c)^{\mathrm{T}} + (\mathbf{Q}_r \otimes \mathbf{Q}_c)(\Psi_r \otimes \mathbf{I}_c)(\mathbf{Q}_r \otimes \mathbf{Q}_c)^{\mathrm{T}}$$

$$= (\mathbf{Q}_r \otimes \mathbf{Q}_c)(\mathbf{I}_r \otimes \Psi_c + \Psi_r \otimes \mathbf{I}_c)(\mathbf{Q}_r \otimes \mathbf{Q}_c)^{\mathrm{T}} = \mathbf{Q}\Psi\mathbf{Q}^{\mathrm{T}}.$$

Note that there is no matrix $\mathbf{Q}$ that can diagonalize $\mathbf{B}_n^j + \mathbf{B}_n^{j\mathrm{T}}$ for all $1 \le j \le n$. Therefore, we need to consider approximate diagonalization.

5. Approximate Diagonalization of Covariance in Two Dimension

In Whittle's (1954) terminology, we define a two dimensional bilateral scheme as follows. Suppose that we have an $r \times c$ rectangular lattice with observations $\{Y_{t,s}\}$ $t=0,1,\cdots r-1$ and $s=0,1,\cdots c-1$. We can define the covariance Γ in terms of the $\mathbf{B}$'s from (2.1),

$$\Gamma = \gamma(0,0)\mathbf{I}_N + \sum_{j=1}^{c-1}[\gamma(j,0)(\mathbf{I}_c \otimes \mathbf{B}_c^j) + \gamma(-j,0)(\mathbf{I}_c \otimes \mathbf{B}_c^{j\mathrm{T}})] \tag{5.1}$$

$$+ \sum_{j=1}^{r-1}[\gamma(0,j)(\mathbf{B}_r^j \otimes \mathbf{I}_c) + \gamma(0,-j)(\mathbf{B}_r^{j\mathrm{T}} \otimes \mathbf{I}_c)]$$

$$+ \sum_{j=1}^{r-1}\sum_{k=1}^{c-1}[\gamma(k,j)(\mathbf{B}_r^j \otimes \mathbf{B}_c^k) + \gamma(-k,-j)(\mathbf{B}_r^{j\mathrm{T}} \otimes \mathbf{B}_c^{k\mathrm{T}})$$

$$+ \gamma(k,-j)(\mathbf{B}_r^{j\mathrm{T}} \otimes \mathbf{B}_c^k) + \gamma(-k,j)(\mathbf{B}_r^j \otimes \mathbf{B}_c^{k\mathrm{T}})],$$

where $\gamma(k,j) = Cov(Y_{t,s}, Y_{t+k,s+j})$.

The two dimensional Fourier matrix $\mathbf{P}_N$ is defined in (4.4). It diagonalizes the circular covariance matrix Γ_s which is defined in terms of Π's from (2.2),

$$\Gamma_s = \gamma(0,0)\mathbf{I}_N + \sum_{j=1}^{m_c}[\gamma(j,0)(\mathbf{I}_r \otimes \Pi_c^j) + \gamma(-j,0)(\mathbf{I}_r \otimes \Pi_c^{j\mathrm{T}})] \tag{5.2}$$

$$+ \sum_{j=1}^{m_r}[\gamma(0,j)(\Pi_r^j \otimes \mathbf{I}_c) + \gamma(0,-j)(\Pi_r^{j\mathrm{T}} \otimes \mathbf{I}_c)]$$

$$+ \sum_{j=1}^{m_r}\sum_{k=1}^{m_c}[\gamma(k,j)(\Pi_r^j \otimes \Pi_c^k) + \gamma(-k,-j)(\Pi_r^{j\mathrm{T}} \otimes \Pi_c^{k\mathrm{T}})$$

$$+ \gamma(k,-j)(\Pi_r^{j\mathrm{T}} \otimes \Pi_c^k) + \gamma(-k,j)(\Pi_r^j \otimes \Pi_c^{k\mathrm{T}})].$$

The diagonalization of (5.2) is

$$\mathbf{P}_N^* \Gamma_s \mathbf{P}_N = \Lambda = \gamma(0,0)\mathbf{I}_N + \sum_{j=1}^{m_c}[\gamma(j,0)(\mathbf{I}_r \otimes \Lambda_c^j) + \gamma(-j,0)(\mathbf{I}_r \otimes \Lambda_c^{j*})] \tag{5.3}$$

$$+ \sum_{j=1}^{m_r}[\gamma(0,j)(\Lambda_r^j \otimes \mathbf{I}_c) + \gamma(0,-j)(\Lambda_r^{j*} \otimes \mathbf{I}_c)]$$

$$+\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}[\gamma(k,j)(\Lambda_r^j \otimes \Lambda_c^k)+\gamma(-k,-j)(\Lambda_r^{j*} \otimes \Lambda_c^{k*})$$

$$+\gamma(k,-j)(\Lambda_r^{j*} \otimes \Lambda_c^k)+\gamma(-k,j)(\Lambda_r^j \otimes \Lambda_c^{k*})] .$$

Suppose that the covariance is radially symmetric. That is

$\gamma(k,j)=\gamma(-k,-j)$ and $\gamma(k,-j)=\gamma(-k,j)$ for $j=0,1,\cdots r-1$ and $k=0,1,\cdots c-1$.

Let us define

$$\gamma_1(j)=\gamma(j,0)\ ,\ \gamma_2(j)=\gamma(0,j)\ ,\ \gamma_3(k,j)=\gamma(k,j) \text{ and } \gamma_4(k,j)=\gamma(-k,j) . \tag{5.4}$$

Then (5.1) and (5.2) can be written as

$$\Gamma=\gamma(0,0)\mathbf{I}_N+\sum_{j=1}^{c-1}\gamma_1(j)\mathbf{M}_1^{(j)}+\sum_{j=1}^{r-1}\gamma_2(j)\mathbf{M}_2^{(j)} \tag{5.5}$$

$$+\sum_{j=1}^{r-1}\sum_{k=1}^{c-1}[\gamma_3(k,j)\mathbf{M}_3^{(j,k)}+\gamma_4(k,j)\mathbf{M}_4^{(j,k)}] ,$$

$$\Gamma_s=\gamma(0,0)\mathbf{I}_N+\sum_{j=1}^{m_c}\gamma_1(j)\mathbf{W}_1^{(j)}+\sum_{j=1}^{m_r}\gamma_2(j)\mathbf{W}_2^{(j)} \tag{5.6}$$

$$+\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}[\gamma_3(k,j)\mathbf{W}_3^{(j,k)}+\gamma_4(k,j)\mathbf{W}_4^{(j,k)}] .$$

An example of Γ in (5.5) is laid out in Figure B . With this assumption of radial symmetry, the diagonal matrix Λ becomes:

$$\Lambda=\gamma(0,0)\mathbf{I}_N+\sum_{j=1}^{m_c}\gamma_1(j)(\mathbf{I}_r \otimes (\Lambda_c^j+\Lambda_c^{j*}))+\sum_{j=1}^{m_r}\gamma_2(j)((\Lambda_r^j+\Lambda_r^{j*}) \otimes \mathbf{I}_c) \tag{5.7}$$

$$+\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}\left\{\gamma_3(k,j)[(\Lambda_r^j \otimes \Lambda_c^k)+(\Lambda_r^{j*} \otimes \Lambda_c^{k*})]+\gamma_4(k,j)[(\Lambda_r^{j*} \otimes \Lambda_c^k)+(\Lambda_r^j \otimes \Lambda_c^{k*})]\right\},$$

Let us now show the asymptotic diagonalization of Γ defined in (5.5) by $\mathbf{P}_N$ defined in (4.4).

Theorem 1 Suppose that we have a rectangular $r\times c$ lattice from a stationary process and suppose that $\gamma(k,j)=\gamma(-k,-j)$ for $k,j=0,\pm1,\pm2,\cdots$. Suppose that γ is absolutely summable with respect to both indices,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\gamma(k,j)|=g<\infty .$$

Then with $\mathbf{P}_N$, Γ , Γ_s and Λ being defined in (4.4), (5.5), (5.6) and (5.7) respectively,

$$\|\mathbf{P}_N^*(\Gamma_s-\Gamma)\mathbf{P}_N\|\rightarrow 0$$

as $min(r,c)\rightarrow\infty$ and $max(r,c)/min(r,c)\rightarrow$ $constant<\infty$.

Remark. Note that the lattice goes to an infinite field in a fairly even way. The rows and columns go to infinity at the same rate.

Proof. Without loss of generality, let us assume both r and c are odd. The proof is similar for either one or both of r and c even.

$$\Gamma_s - \Gamma = \sum_{j=1}^{m_c} \gamma_1(j)\mathbf{W}_1^{(j)} - \sum_{j=1}^{c-1} \gamma_1(j)\mathbf{M}_1^{(j)}$$

$$+ \sum_{j=1}^{m_r} \gamma_2(j)\mathbf{W}_2^{(j)} - \sum_{j=1}^{r-1} \gamma_2(j)\mathbf{M}_2^{(j)}$$

$$+ \sum_{j=1}^{m_r}\sum_{k=1}^{m_c} [\gamma_3(k,j)\mathbf{W}_3^{(j,k)} + \gamma_4(k,j)\mathbf{W}_4^{(j,k)}] - \sum_{j=1}^{r-1}\sum_{k=1}^{c-1} [\gamma_3(k,j)\mathbf{M}_3^{(j,k)} + \gamma_4(k,j)\mathbf{M}_4^{(j,k)}]$$

$$= \sum_{j=1}^{m_c} [\gamma_1(j) - \gamma_1(c-j)]\mathbf{M}_1^{(c-j)} + \sum_{j=1}^{m_r} [\gamma_2(j) - \gamma_2(r-j)]\mathbf{M}_2^{(r-j)}$$

$$+ \sum_{j=1}^{m_r}\sum_{k=1}^{m_c}\sum_{l=3}^{4} \Big\{ [\gamma_l(k,j) - \gamma_l(c-k,r-j)]\mathbf{M}_l^{(r-j,c-k)} + [\gamma_l(k,j) - \gamma_l(k,r-j)]\mathbf{M}_l^{(r-j,k)}$$

$$+ [\gamma_l(k,j) - \gamma_l(c-k,j)]\mathbf{M}_l^{(j,c-k)} \Big\}.$$

Note that the diagonal elements of $\mathbf{P}_N^*(\Gamma_s - \Gamma)\mathbf{P}_N$ are larger than the off-diagonal elements on the same row. That is

$$|\mathbf{P}_{.\mu}^* (\Gamma_s - \Gamma)\mathbf{P}_{.\xi}| \le |\mathbf{P}_{.\mu}^* (\Gamma_s - \Gamma)\mathbf{P}_{.\mu}| \quad \text{for} \quad \mu \ne \xi$$

Thus it suffices to show that $|\mathbf{P}_{.\mu}^* (\Gamma_s - \Gamma)\mathbf{P}_{.\mu}| \to 0$ as r and $c \to \infty$ in the sense stated in the theorem.

$$|\mathbf{P}_{.\mu}^* (\Gamma_s - \Gamma)\mathbf{P}_{.\mu}|$$

$$= \Big| \sum_{j=1}^{m_c} [\gamma_1(j) - \gamma_1(c-j)] \frac{2j}{c} cos(\frac{2\pi}{c} j v_\mu) + \sum_{j=1}^{m_r} [\gamma_2(j) - \gamma_2(c-j)] \frac{2j}{r} cos(\frac{2\pi}{c} j u_\mu) \tag{5.8}$$

$$+ \sum_{j=1}^{m_r}\sum_{k=1}^{m_c} \Big\{ 2cos(\frac{2\pi}{r} j u_\mu + \frac{2\pi}{c} k v_\mu) \times$$

$$[\frac{(r-j)k}{rc}(\gamma_4(k,j) - \gamma_4(c-k,j)) + \frac{j(c-k)}{rc}(\gamma_4(k,j) - \gamma_4(k,r-j)) + \frac{jk}{rc}(\gamma_3(k,j) - \gamma_3(c-k,r-j))]$$

$$+ 2cos(\frac{2\pi}{r} j u_\mu - \frac{2\pi}{c} k v_\mu) \times$$

$$[\frac{(r-j)k}{rc}(\gamma_3(k,j) - \gamma_3(c-k,j)) + \frac{j(c-k)}{rc}(\gamma_3(k,j) - \gamma_3(k,r-j)) + \frac{jk}{rc}(\gamma_4(k,j) - \gamma_4(c-k,r-j))] \Big\} \Big|$$

$$\le \sum_{j=1}^{m_c} \frac{2j}{c} |\gamma_1(j) - \gamma_1(c-j)| + \sum_{j=1}^{m_r} \frac{2j}{r} |\gamma_2(j) - \gamma_2(r-j)| \tag{5.9}$$

$$+ \sum_{j=1}^{m_r}\sum_{k=1}^{m_c}\sum_{l=3}^{4} [2 \frac{(r-j)k}{rc} |\gamma_l(k,j) - \gamma_l(c-k,j)|$$

$$+2\frac{j(c-k)}{rc}|\gamma_l(k,j)-\gamma_l(k,r-j)|+2\frac{jk}{rc}|\gamma_l(k,j)-\gamma_l(c-k,r-j)|].$$

To show the bound in (5.9) goes to zero, we need the following lemma.

Lemma 1 (Two Dimensional Kronecker Lemma). Let $\{a_i\}$ and $\{b_j\}$ be two increasing sequences of positive numbers. Let $\{X_{ij}\}$ be a lattice sequence of non-negative real numbers, which satisfy

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}X_{ij}=X<\infty,$$

which implies

$$\sum_{i=1}^{\infty}X_{ij}=X_{.j}<\infty \quad \text{for } all\ j \quad \text{and} \quad \sum_{j=1}^{\infty}X_{ij}=X_{i.}<\infty \quad \text{for } all\ i\,.$$

Then

$$\sum_{i=1}^{r}\sum_{j=1}^{c}\frac{a_i}{a_r}\frac{b_j}{b_c}X_{ij}\to 0\,,\ \sum_{i=1}^{r}\frac{a_i}{a_r}X_{ij}\to 0 \quad \text{for } all\ j \text{ and } \sum_{j=1}^{c}\frac{b_j}{b_c}X_{ij}\to 0 \quad \text{for } all\ i$$

as r and $c\to\infty$.

Proof of the lemma.

Let $S_{r,c}=\sum_{i=1}^{r}\sum_{j=1}^{c}X_{ij}$ be the finite sums. Then

$$\sum_{i=1}^{r}\sum_{j=1}^{c}\frac{a_i}{a_r}\frac{b_j}{b_c}X_{ij}=\sum_{i=1}^{r}\sum_{j=1}^{c}\frac{a_i}{a_r}\frac{b_j}{b_c}(S_{i,j}-S_{i,j-1}-S_{i-1,j}+S_{i-1,j-1})$$

$$=S_{r,c}-S_{r,c-1}-S_{r-1,c}+S_{r-1,c-1}+\sum_{j=1}^{c-1}\frac{b_j}{b_c}[S_{r,j}-S_{r-1,j}-S_{r,j-1}+S_{r-1,j-1}]$$

$$+\sum_{i=1}^{r-1}\frac{a_i}{a_r}[S_{i,c}-S_{i-1,c}-S_{i,c-1}+S_{i-1,c-1}]+\sum_{i=1}^{r-1}\sum_{j=1}^{c-1}\frac{a_i}{a_r}\frac{b_j}{b_c}[S_{i,j}-S_{i,j-1}-S_{i-1,j}+S_{i-1,j-1}]$$

$$=S_{r,c}-S_{r,c-1}-S_{r-1,c}+\sum_{i=0}^{r-1}\sum_{j=0}^{c-1}\frac{(a_{i+1}-a_i)(b_{j+1}-b_j)}{a_r b_c}S_{i,j}\,. \tag{5.10}$$

Note that $\{a_i\}$ and $\{b_j\}$ are increasing, so both $(a_{i+1}-a_i)$ and $(b_{j+1}-b_j)$ are positive. Without loss of generality, let $S_{0,j}=S_{i,0}=S_{0,0}=0$ and $a_0=b_0=0$. Since the $S_{i,j}\to X$ as i and j go to infinity, the expression in (5.10) goes to zero as r and c go to infinity. Because the X_{ij} are non-negative, the convergence of the marginal sums to zero follows.

□

By applying the above lemma with $X_{jk}=|\gamma(j,k)|$, it is obvious that

$$\sum_{j=1}^{m_c}\frac{2j}{c}|\gamma_1(j)-\gamma_1(c-j)|\,,\ \sum_{j=1}^{m_r}\frac{2j}{r}|\gamma_2(j)-\gamma_2(r-j)|$$

$$\text{and}\ \sum_{j=1}^{m_r}\sum_{k=1}^{m_c}\sum_{l=3}^{4}[2\frac{jk}{rc}|\gamma_l(k,j)-\gamma_l(c-k,r-j)|]$$

go to zero because of the absolute summability of γ. To show that

$$\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}\sum_{l=3}^{4}[2\frac{(r-j)k}{rc}\,|\gamma_l(k,j)-\gamma_l(c-k,j)|+2\frac{j(c-k)}{rc}\,|\gamma_l(k,j)-\gamma_l(k,r-j)|]\rightarrow 0\ ,$$

we examine one of these terms.

$$\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}2\frac{(r-j)k}{rc}\,|\gamma_3(k,j)-\gamma_3(c-k,j)|\le\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}\ [2\frac{(r-j)k}{rc}\,|\gamma_3(k,j)|+2\frac{(r-j)k}{rc}\,|\gamma_3(c-k,j)|\ ]$$

$$=\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}\ [\frac{2k}{c}\,|\gamma_3(k,j)|-\frac{2jk}{rc}\,|\gamma_3(k,j)|+\frac{2k}{c}\,|\gamma_3(c-k,j)|-\frac{2jk}{rc}\,|\gamma_3(c-k,j)|\ ]$$

$$\le\sum_{k=1}^{m_c}[\frac{2k}{c}g_k]+\sum_{k=1}^{m_c}[\frac{2k}{c}g_{(c-k)}]-e_1-e_2$$

where e_1 and e_2 are the double summations with the weights $\frac{jk}{rc}$, which go to zero by Lemma 1, and $g_k=\sum_{j=1}^{\infty}|\gamma(k,j)|<\infty$. Applying the one-dimensional Kronecker Lemma to terms involving g_k gives the zero convergence result. Hence, the proof is completed for r and c odd.

Note that if either one or both of r and c is even, replace m_r or m_c in (5.8) and (5.9) by m_r-1 or m_c-1. Then the proof can still be carried through.

□

Corollary 1

Under the same assumptions as Theorem 1, the convergent rate is of $O(1/q)$ with $q=min(r,c)$.

Proof

From the absolute summability of γ, we have

$$|\gamma(j,k)|=O(1/rc)$$

Then let us examine the upper bound in (5.9)

$$\sum_{j=1}^{m_c}\frac{2j}{c}\,|\gamma_1(j)-\gamma_1(c-j)|=O(1/rc)\ \text{and}\ \sum_{j=1}^{m_r}\frac{2j}{r}\,|\gamma_2(j)-\gamma_2(r-j)|=O(1/rc)$$

For $l=3,4$,

$$\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}[2\frac{jk}{rc}\,|\gamma_l(k,j)-\gamma_l(c-k,r-j)|]=\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}O(1/r)O(1/c)O(1/rc)=O(1/rc)$$

and

$$\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}[2\frac{(r-j)k}{rc}\,|\gamma_l(k,j)-\gamma_l(c-k,j)|]=\sum_{j=1}^{m_r}\sum_{k=1}^{m_c}(1-O(1/r))O(1/c)O(1/rc)=O(1/c)$$

The bound becomes

$$O(1/rc)+O(1/rc)+\sum_{l=3}^{4}[\,O(1/r)+O(1/c)+O(1/rc)]$$

which is dominated by $O(1/q)$ with $q=min(r,c)$.

□

Corollary 2
Suppose that we have the same assumptions as Theorem 1, except that we do not assume radial symmetry. Then

$$\| \mathbf{P}_N^*(\Gamma_s - \Gamma)\mathbf{P}_N \| = O(1/min(r,c)) .$$

Proof We can break down the proof into four parts. One for each index quadrant:$(\pm k, \pm j)$. In each quadrant, we can use the argument in the proof of Theorem 1. Since we have only four partitions, the combined result also holds. □

6. Extension to Three and Higher Dimension

Extending to higher dimension is not very difficult but it is laborious. Based on the building blocks (2.1), (2.2), (2.4) and (2.5), we write down the covariance of a stationary three dimensional process. Using the same notation as before, let us define the covariance of a $r \times c \times l$ lattice as Γ,

$$\Gamma = \sum_{x=-(r-1)}^{r-1} \sum_{y=-(c-1)}^{c-1} \sum_{z=-(l-1)}^{l-1} \gamma(x,y,z)[\mathbf{J}(x,r) \otimes \mathbf{J}(y,c) \otimes \mathbf{J}(z,l)] \tag{6.1}$$

where

$$\gamma(x,y,z) = Cov(Y_{t,s,v} Y_{t+x,s+y,v+z}) \text{ and } x=0,1,\cdots r-1; y=0,1,\cdots c-1; z=0,1,\cdots l-1 .$$

The Fourier matrix that asymptotically diagonalizes this Γ matrix is

$$\mathbf{P}_N = \mathbf{P}_r \otimes \mathbf{P}_c \otimes \mathbf{P}_l , N=r \times c \times l$$

with

$$\{\mathbf{P}_N\}_{jk} = \frac{1}{\sqrt{rcl}} \exp[i(\frac{2\pi}{r} u_j u_k + \frac{2\pi}{c} v_j v_k + \frac{2\pi}{l} w_j w_k)]$$

where $j=u_j cl + v_j l + w_j$ $u_j = 0,1,\cdots r-1$, $v_j = 0,1,\cdots c-1$, $w_j = 0,1,\cdots l-1$.
Hence, the indices j and k are determined in an analogous manner to (4.1). The circular counterpart of Γ, Γ_s has the **J**'s in (6.1) replaced by the **F**'s defined in (2.5) and the summations over m_r, m_c and m_l respectively. With the assumption of absolute summability on γ, the result on asymptotic diagonalization can be extended to the three dimensional case. The details are tedious but straight forward. This result can further be extended to higher dimensions.

If γ is radially symmetric

$$\gamma(x,y,z) = \gamma(-x,-y,-z) \quad \text{for } x,y,z=0,\pm1,\pm2,\cdots$$

then (6.1) can be written as

$$\Gamma = \gamma(0,0,0)\mathbf{I}_N + \sum_{z=1}^{l-1} \gamma(0,0,z)(\mathbf{I}_r \otimes \mathbf{I}_c \otimes (\mathbf{B}_l^z + \mathbf{B}_l^{z\mathrm{T}})) \tag{6.2}$$

$$+ \sum_{y=1}^{c-1} \gamma(0,y,0)(\mathbf{I}_r \otimes (\mathbf{B}_c^y + \mathbf{B}_c^{y\mathrm{T}}) \otimes \mathbf{I}_l)$$

$$+ \sum_{x=1}^{r-1} \gamma(x,0,0)((\mathbf{B}_r^x + \mathbf{B}_r^{x\mathrm{T}}) \otimes \mathbf{I}_c \otimes \mathbf{I}_l)$$

$$+ \sum_{y=1}^{c-1} \sum_{z=1}^{l-1} \gamma(0,y,z)\mathbf{I}_r \otimes (\mathbf{B}_c^y \otimes \mathbf{B}_l^z + \mathbf{B}_c^{y\mathrm{T}} \otimes \mathbf{B}_l^{z\mathrm{T}}) + \gamma(0,y,-z)\mathbf{I}_r \otimes (\mathbf{B}_c^y \otimes \mathbf{B}_l^{z\mathrm{T}} + \mathbf{B}_c^{y\mathrm{T}} \otimes \mathbf{B}_l^z)$$

$$+\sum_{x=1}^{r-1}\sum_{z=1}^{l-1}\gamma(x,0,z)(\mathbf{B}_r^x\otimes\mathbf{I}_c\otimes\mathbf{B}_l^z+\mathbf{B}_r^{x\mathrm{T}}\otimes\mathbf{I}_c\otimes\mathbf{B}_l^{z\mathrm{T}})+\gamma(x,0,-z)(\mathbf{B}_r^x\otimes\mathbf{I}_c\otimes\mathbf{B}_l^{z\mathrm{T}}+\mathbf{B}_r^{x\mathrm{T}}\otimes\mathbf{I}_c\otimes\mathbf{B}_l^z)$$

$$+\sum_{x=1}^{r-1}\sum_{y=1}^{c-1}\gamma(x,y,0)(\mathbf{B}_r^x\otimes\mathbf{B}_c^y+\mathbf{B}_r^{x\mathrm{T}}\otimes\mathbf{B}_c^{y\mathrm{T}})\otimes\mathbf{I}_l+\gamma(x,-y,0)(\mathbf{B}_r^x\otimes\mathbf{B}_c^{y\mathrm{T}}+\mathbf{B}_r^{x\mathrm{T}}\otimes\mathbf{B}_c^y)\otimes\mathbf{I}_l$$

$$+\sum_{x=1}^{r-1}\sum_{y=1}^{c-1}\sum_{z=1}^{l-1}[\gamma(x,y,z)(\mathbf{B}_r^x\otimes\mathbf{B}_c^y\otimes\mathbf{B}_l^z+\mathbf{B}_r^{x\mathrm{T}}\otimes\mathbf{B}_c^{y\mathrm{T}}\otimes\mathbf{B}_l^{z\mathrm{T}})$$

$$+\gamma(x,y,-z)(\mathbf{B}_r^x\otimes\mathbf{B}_c^y\otimes\mathbf{B}_l^{z\mathrm{T}}+\mathbf{B}_r^{x\mathrm{T}}\otimes\mathbf{B}_c^{y\mathrm{T}}\otimes\mathbf{B}_l^z)]$$

$$+\sum_{x=1}^{r-1}\sum_{y=1}^{c-1}\sum_{z=1}^{l-1}[\gamma(x,-y,z)(\mathbf{B}_r^x\otimes\mathbf{B}_c^{y\mathrm{T}}\otimes\mathbf{B}_l^z+\mathbf{B}_r^{x\mathrm{T}}\otimes\mathbf{B}_c^y\otimes\mathbf{B}_l^{z\mathrm{T}})$$

$$+\gamma(-x,y,z)(\mathbf{B}_r^{x\mathrm{T}}\otimes\mathbf{B}_c^y\otimes\mathbf{B}_l^z+\mathbf{B}_r^x\otimes\mathbf{B}_c^{y\mathrm{T}}\otimes\mathbf{B}_l^{z\mathrm{T}})]\,.$$

For any finite d-dimensional lattice, the above formulation of Γ can be extended by using the building blocks and the $\mathbf{J}$ matrices. Suppose that we have a d-dimensional lattice with edge sizes $n_1, \cdots, n_d$. Then we define the non-circular covariance as

$$\Gamma=\sum_{j_1=-(n_1-1)}^{n_1-1}\cdots\sum_{j_d=-(n_d-1)}^{n_d-1}\gamma(j_1,\cdots,j_d)[\mathbf{J}(j_1,n_1)\otimes\cdots\otimes\mathbf{J}(j_d,n_d)] \tag{6.3}$$

and its circular counterpart as

$$\Gamma_s=\sum_{j_1=-m_1}^{m_1}\cdots\sum_{j_d=-m_d}^{m_d}\gamma(j_1,\cdots,j_d)[\mathbf{F}(j_1,n_1)\otimes\cdots\otimes\mathbf{F}(j_d,n_d)] \tag{6.4}$$

where the $\mathbf{J}$'s and the $\mathbf{F}$'s are defined in (2.4) and (2.5), respectively and $m_i=[n_i/2]$. Furthermore, $\gamma(j_1,\cdots,j_d)=Cov(Y_{t_1},\ldots,_{t_d}Y_{t_1+j_1},\ldots,_{t_d+j_d})$. Absolute summability assumption on γ is needed, i.e.,

$$\sum_{j_1=-\infty}^{\infty}\cdots\sum_{j_d=-\infty}^{\infty}|\gamma(j_1,\cdots,j_d)|=constant<\infty\,.$$

The asymptotic diagonalization of Γ in (6.3) can be extended to d-dimensional using the d-dimensional Kronecker Lemma.

Lemma 2. Suppose that

$$\sum_{j_1=1}^{\infty}\cdots\sum_{j_d=1}^{\infty}X_{j_1},\ldots,_{j_d}=X<\infty$$

and $\{a_{j_1}\},\cdots,\{a_{j_d}\}$ are d increasing sequences of positive numbers, Then

$$\sum_{j_1=1}^{n_1}\cdots\sum_{j_d=1}^{n_d}\frac{a_{j_1}}{a_{n_1}}\cdots\frac{a_{j_d}}{a_{n_d}}X_{j_1},\ldots,_{j_d}\to 0$$

as $n_1,\cdots,n_d\to\infty$ in the sense that $\dfrac{min(n_1,\cdots,n_d)}{max(n_1,\cdots,n_d)}<\infty$.

Remark The proof of this lemma is a straightforward extension of the proof of Lemma 1. The basic steps are the same but the number of partial sums involved is 2^d. From this we can deduce the asymptotic diagonalization of a covariance of a d-dimensional lattice. Hence, the rate of convergence is of $O(1/q)$ where $q=min(n_1,\cdots,n_d)$. A recent piece of work on asymptotic likelihood estimation in d-dimensional Gaussian lattice is given by Guyon

(1982) .

7. Examples and Applications

Let us consider the two-dimensional case. The covariance of (5.1) is quite general. It requires only stationarity and diminishing autocorrelation (absolute summability). The conditional AR and the simultaneous AR processes on the lattice are two common examples that satisfy the above theorem.

Examples

(1) Conditional AR(1) on an $r \times c$ grid with different parameters for the vertical and horizontal correlations:

$$E(Y_{t,s} \mid all\ Y's\ ) = \rho_1(Y_{t,s-1}+Y_{t,s+1}) + \rho_2(Y_{t-1,s}+Y_{t+1,s})$$

where ρ_1 measures the row (east-west) autocorrelation and ρ_2 measures the column (north-south) autocorrelation. Its covariance can be expressed as

$$\gamma(k,j) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos(k\theta_1)\cos(j\theta_2)}{1-2\rho_1\cos(\theta_1)-2\rho_2\cos(\theta_2)}\partial\theta_1\partial\theta_2$$

Note that $\gamma(k,j)=\gamma(-k,-j)$ and $\gamma(k,-j)=\gamma(-k,j)$ but $\gamma(j,0)\neq\gamma(0,j)$ if $\rho_1\neq\rho_2$.

(2) The simultaneous AR(1) with one parameter:

$$Y_{ts} = \rho(Y_{t,s-1}+Y_{t,s+1}+Y_{t-1,s}+Y_{t+1,s}) + \ white\ noise$$

where the variance of the white noise is σ^2 and the autoregressive parameter is ρ. Its covariance is

$$\gamma(k,j) = \frac{\sigma^2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{\cos(k\theta_1)\cos(j\theta_2)}{[\ 1-2\rho(\cos(\theta_1)+\cos(\theta_2))\ ]^2}\partial\theta_1\partial\theta_2$$

Note that if $r=c$ then $\gamma(j,k)=\gamma(k,j)$ in addition to radial symmetry for the conditional AR model.

Applications

(1) In spectral analysis of time series, the power spectra are asymptotically uncorrelated (Priestley, 1984, p.405). The use of two-dimensional spectral analysis has been illustrated in Cliff and Ord (1981) , Ripley (1981) and Renshaw (1984). Because the orthogonal matrix $\mathbf{P}_N$ we use to diagonalize a circular neighborhood matrix is a Fourier matrix, the square of each component of $\mathbf{Z}=\mathbf{P}_N^*\mathbf{Y}$ is the sample periodogram, where $\mathbf{Y}$ is the observation vector. Suppose that j is the d-dimensional analogue of j in (4.1). Then $\mathbf{Z}_j{}^2$ is the periodogram at the d-tuple frequency associated with j. Therefore, the asymptotic uncorrelation among periodograms of higher dimensional lattice is the consequence of our current discussion.

(2) Field trial experiments:

This asymptotic diagonalization can be used in field trial experiments to remove spatial dependency among agricultural plots. For example, the matrix $\mathbf{P}_N$ in (4.4) can be used to aid the estimation of parameters from modelling general fertility or local competition on a field. Suppose that we have

$(i)\ \mathbf{Y} = \mathbf{D}\tau + \xi + \varepsilon$ *(soil fertility trend model)*

or

$(ii)\ (\mathbf{I}-\rho\mathbf{M})\mathbf{Y} = \mathbf{D}\tau + \varepsilon$ *(local competition model)*.

In both cases, $\mathbf{D}$ represents the design matrix and τ the corresponding effect. ε is the white noise with variance σ_ε^2 and mean zero. ξ is a random variable representing the general fertility effect which might have a spatial structure depicted in $V(\xi)$. $\mathbf{M}$ is the immediate or nearest-neighborhood matrix (non-circular) where ρ is the autocorrelation parameter.

Let $\mathbf{Z} = \mathbf{P}_N^* \mathbf{Y}$. Then

$$(i)\ Var(\mathbf{Z}) = \mathbf{P}_N^* V(\xi)\mathbf{P}_N + \sigma_\varepsilon^2 \mathbf{I}$$

and

$$(ii)\ Var(\mathbf{Z}) = \sigma_\varepsilon^2 \mathbf{P}_N^* (\mathbf{I}-\rho\mathbf{M})^{-1}(\mathbf{I}-\rho\mathbf{M})^{-T}\mathbf{P}_N \ .$$

Both covariance matrices are approximately diagonal for a large lattice.

In both cases, the use of $\mathbf{P}_N$ to diagonalize the covariance is demonstrated in the estimation of the autocorrelation parameter ρ. The approximate diagonalization reduces the computational task because no decomposition of an $N \times N$ matrix is required.

Acknowledgements

This research was supported by United States Department of Agriculture CSRS grant 511-100. Computing was performed on the UW-Madison Statistics VAX 11/750 Research Computer.

References

Cliff, A.D. and Ord, J.K. (1981) *Spatial Processes Models and Applications.* London: Pion.

Conte, S.D. and deBoor, Carl (1980) *Elementary numerical analysis: an algorithmic approach.* New York: McGraw-Hill Book Company.

Davis, Philip J. (1979) *Circulant Matrices.* New York: John Wiley and Sons.

Fuller, Wayne (1976) *Introduction to Statistical Time Series.* New York: John Wiley and Sons.

Guyon, Xavier (1982) Parameter estimation for a stationary process on a d-dimensional lattice. *Biometrika,* **69(1)**, 95-105.

Moran, P. A. P. (1973a) A Gaussian Markovian process on a square lattice. *Journal of Applied Probability,* **10**, 54-62.

Moran, P. A. P. (1973b) Necessary conditions for Markovian processes on a lattice. *Journal of Applied Probability,* **10**, 605-612.

Priestley, M.B. (1984) *Spectral Analysis and Time Series vol. 1 & 2.* London: Academic Press.

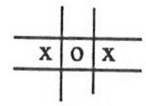
Renshaw, Eric (1984) Competition experiments for light in a plant monoculture: An analysis based on two-dimensional spectra. *Biometrics,* **40**, 717-728.

Ripley, B. D. (1981) *Spatial Statistics.* New York: John Wiley & Sons.

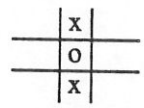
Whittle, P. (1954) On stationary processes in the plane. *Biometrika,* **41**, 434-449.

Figure A

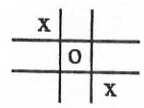
W_1 associates with neighbors 'x' in



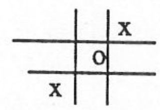
W_2 associates with neighbors 'x' in



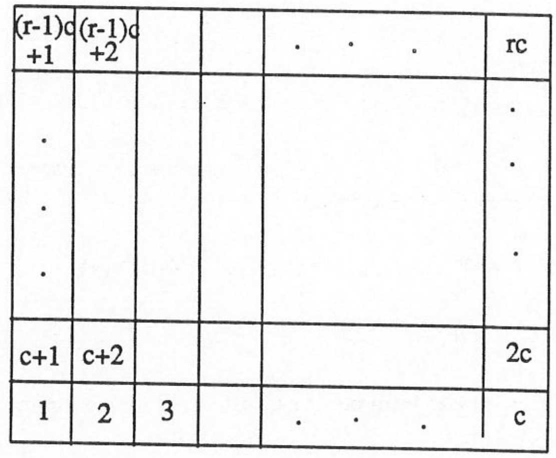
W_3 associates with neighbors 'x' in



W_4 associates with neighbors 'x' in



Label the lattice as



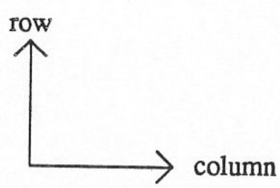


Figure B

An example of Γ

For r=4 and c=5, let us write out the Γ matrix defined in (5.5). Let us define

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & \Gamma_{34} \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} \end{bmatrix}$$

where $\Gamma_{ij} = \Gamma_{ji}$.

$$\Gamma_{ii} = \begin{bmatrix} \gamma(0,0) & \gamma_1(1) & \gamma_1(2) & \gamma_1(3) & \gamma_1(4) \\ & \gamma(0,0) & \gamma_1(1) & \gamma_1(2) & \gamma_1(3) \\ & & \gamma(0,0) & \gamma_1(1) & \gamma_1(2) \\ & & & \gamma(0,0) & \gamma_1(1) \\ & & & & \gamma(0,0) \end{bmatrix} \quad \text{for } i=1,\cdots,4.$$

$$\Gamma_{i,j+1} = \begin{bmatrix} \gamma_2(1) & \gamma_3(2) & \gamma_3(3) & \gamma_3(4) & \gamma_3(5) \\ \gamma_4(2) & \gamma_2(1) & \gamma_3(2) & \gamma_3(3) & \gamma_3(4) \\ \gamma_4(3) & \gamma_4(2) & \gamma_2(1) & \gamma_3(2) & \gamma_3(3) \\ \gamma_4(4) & \gamma_4(3) & \gamma_4(2) & \gamma_2(1) & \gamma_3(2) \\ \gamma_4(5) & \gamma_4(4) & \gamma_4(3) & \gamma_4(2) & \gamma_2(1) \end{bmatrix} \quad \text{for } i=1,\cdots,3.$$

$$\Gamma_{i,j+2} = \begin{bmatrix} \gamma_2(2) & \gamma_3(3) & \gamma_3(4) & \gamma_3(5) & \gamma_3(6) \\ \gamma_4(3) & \gamma_2(2) & \gamma_3(3) & \gamma_3(4) & \gamma_3(5) \\ \gamma_4(4) & \gamma_4(3) & \gamma_2(2) & \gamma_3(3) & \gamma_3(4) \\ \gamma_4(5) & \gamma_4(4) & \gamma_4(3) & \gamma_2(2) & \gamma_3(3) \\ \gamma_4(6) & \gamma_4(5) & \gamma_4(4) & \gamma_4(3) & \gamma_2(2) \end{bmatrix} \quad \text{for } i=1,2.$$

$$\Gamma_{14} = \begin{bmatrix} \gamma_2(3) & \gamma_3(4) & \gamma_3(5) & \gamma_3(6) & \gamma_3(7) \\ \gamma_4(4) & \gamma_2(3) & \gamma_3(4) & \gamma_3(5) & \gamma_3(6) \\ \gamma_4(5) & \gamma_4(4) & \gamma_2(3) & \gamma_3(4) & \gamma_3(5) \\ \gamma_4(6) & \gamma_4(5) & \gamma_4(4) & \gamma_2(3) & \gamma_3(4) \\ \gamma_4(7) & \gamma_4(6) & \gamma_4(5) & \gamma_4(4) & \gamma_2(3) \end{bmatrix}.$$