Review

1. The mean and variance of X$\sim$ Binomial(n,p) are $E(X) = np, \quad Var(X) = np(1-p)$.

2. If X_1, X_2, ..., X_n constitute a random sample from a distribution with mean μ and variance σ^2, then:
$$E(\bar{X}) = \mu \text{ and } Var(\bar{X}) = \sigma^2/n$$
$$E(\sum_{i=1}^{n} X_i) = n\mu \text{ and } Var(\sum_{i=1}^{n} X_i) = n\sigma^2$$

3. Let random sample $X_1, X_2, \cdots, X_n \sim N(\mu, \sigma^2)$, then random variables $\bar{X}$ and $\sum_{i=1}^n X_i$ are distributed as
$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \text{ and } \sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2).$$

4. **Central Limit Theorem**
For a random sample $X_1, X_2, \cdots, X_n$ (when n is large) from an arbitrary distribution with mean μ and variance σ^2, $\bar{X}$ will be approximately distributed by $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

5. Let $X \sim B(n,p)$, then the normal approximation for X is
$$X \sim N(np, np(1-p)).$$
The proportion $Y = \frac{X}{n}$ can be approximated by a normal distribution $N(p, p(1-p)/n)$.
The requirement is that $np \geq 5$ and $n(1-p) \geq 5$ for both approximations.

6. Let random sample $X_1, X_2, \cdots, X_n \sim N(\mu, \sigma^2)$, then random variable
$$V^2 = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1},$$
where S^2 is the sample variance defined as $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

Practice Problem

1. Suppose the random variable X is distributed as $N(100, 144)$.
   (a) Consider a random sample with $n = 16$, what is the distribution of $\bar{X} = \frac{1}{16}\sum_{i=1}^{16} X_i$?
   (b) Find out $\Pr(\bar{X} < 103)$.
   (c) What is the distribution of $\sum_{i=1}^{16} X_i$?
   (d) How large does n have to be in order that $Var(\bar{X}) = 1$?

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2. Suppose that 20% of the trees in a forest are infected with a certain type of parasite. Let X denote the number of trees having the parasite in a random sample of 300 trees.

   (a) Compute $\Pr(49 \leq X \leq 71)$.

   (b) Define Y to be the proportion of trees having parasite: $Y = X/300$.Find $\Pr(Y < 0.25)$.

3. Suppose we have observations from a $N(\mu, \sigma^2)$ distribution. Compute the following:

μ	σ^2	n	a	$\Pr(S^2 \leq a)$	b	$Pr(S^2 \geq b)$
-24	20	11	7	_____	9	_____
250	7	29	_____	.025	_____	.99
6	_____	8	8.5	.95		

4. There are two coins, a green one and a red one. The green coin is fair(probability of heads is 0.5), whereas the red coin has probability of heads equal to 0.8.

   (a) Suppose the green coin is flipped independently twice and the red coin is flipped once. Assume that the flips of the green and red coins are independent of each other. Find the probability that the *total* number of heads is exactly 2.

   (b) Let X be the total number of heads from the 4 flips. Find the probability distribution of X.

   (c) Find the expected value of X and the variance of X.

Solution for practice problem

1. (a) $\bar{X} \sim N(100, \frac{144}{16})$, that is, $\bar{X} \sim N(100, 9)$.

   (b) $\Pr(\bar{X} < 103)$=$\Pr(Z < \frac{103-100}{3})$=$\Pr(Z < 1)$=0.8413.

   (c) $\sum_{i=1}^{16} X_i \sim N(16(100), 16(144)) = N(1600, 2304)$.

   (d) $Var(\bar{X}) = \sigma^2/n = \frac{144}{n}$. sample size n has to be 144 in order that $Var(\bar{X}) = 1$.

2. (a) $X \sim Bin(300, 0.20)$. Approximately, $X \sim N(300(.20), 300(.20)(.80)) = N(60, 48)$. Therefore
   $\Pr(49 \le X \le 71)$=$\Pr(\frac{49-60}{\sqrt{48}} < Z < \frac{71-60}{\sqrt{48}})$
   =$\Pr(-1.59 \le Z \le 1.59) = 1 - 2(.0559) = .8882$.

   (b) $Y = \frac{X}{300} \sim N((.20, \frac{.20(.80)}{300}) = N(.20, .00053)$,
   $\Pr(Y < 0.25) = \Pr(Z < \frac{0.25-0.20}{0.023}) = \Pr(Z < 2.17) = 1 - .0150$ =.9850.

3. (a) (first row)
   $\Pr(S^2 \le 7) = \Pr(V^2 \le \frac{(11-1)7}{20}) = \Pr(V^2 \le 3.5)$, 0.025< $\Pr(S^2 \le 7)$<0.05,
   $\Pr(S^2 \ge 9) = \Pr(V^2 \ge \frac{(11-1)9}{20}) = \Pr(V^2 \ge 4.5)$, 0.90< $\Pr(S^2 < 7)$<0.95,

   (b) (second row)
   $\Pr(S^2 \le a) = \Pr(V^2 \le \frac{(29-1)a}{7}) = 0.025$, $\frac{(29-1)a}{7}$=15.31, a=3.83.
   $\Pr(S^2 \ge b) = \Pr(V^2 \le \frac{(29-1)b}{7}) = 0.99$, $\frac{(29-1)b}{7}$=13.56, b=3.39.

   (c) (third row)
   $\Pr(S^2 \le 8.5) = \Pr(V^2 \le \frac{(8-1)8.5}{\sigma^2}) = 0.95$, $\frac{(8-1)8.5}{\sigma^2}$=14.07, σ^2=4.23.

4. Define X_1 as the number of heads obtained from the green coin and X_2 as the number of heads for the red coin. Then X_1, X_2 are independent and we have the probability distributions of $X - 1$ and X_2:

X_1	0	1	2
$P(x_1)$	.25	.5	.25

X_2	0	1
$P(x_2)$	.2	.8

   (a) $P(X = 2) = P(X_1 = 1, X_2 = 1) + P(X_1 = 2, X_2 = 0) = (.5)(.8) + (.25)(.2) = .45$.

   (b)

X	0	1	2	3
P(x)	.05	.3	.45	.2

   (c) E(X)=1.8, which is the same as $E(X_1) + E(X_2) = 1 + 0.8$,
   Var(X)=0.82, which equals to $Var(X_1) + Var(X_2) = 0.5 + 0.32$.