Empirical Distribution of Equilibrium Play and Its Testing Application

Yakov Babichenko, Siddharth Barman, and Ron Peretz

NEGT
Goal: Test if players are implementing an equilibrium using observed behavior
Goal: Test if players are implementing an equilibrium using observed behavior in large games

- Players: $[n]$
- Actions of each player: $[m]$
Goal: Test if players are implementing a Nash equilibrium using observed behavior in large games

- Players: \([n]\)
- Actions of each player: \([m]\)
- Mixed strategy of player \(i\): \(x_i \in \Delta^m\)
Goal: Test if players are implementing a Nash equilibrium using i.i.d. samples in large games

- Players: $[n]$
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- Mixed strategy of player $i$: $x_i \in \Delta^m$
- Data: i.i.d samples $\sim x_i$ for each player $i$
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✓ Feasible (with small slack)

Q: How many samples are sufficient?
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Q: How many samples are sufficient?

Our Result (informally)

A very small number of samples are sufficient for testing if players are implementing a Nash equilibrium.
Goal: Test if players are implementing a Nash equilibrium using i.i.d. samples in large games

- Players: \([n]\)
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Q: How many samples are sufficient?

Our Result (informally)

\(O(\log m + \log n)\) samples are sufficient for testing if players are implementing a Nash equilibrium.
Goal: Test if players are implementing a Nash equilibrium using i.i.d. samples in large games

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- Actions of each player: \([m]\)
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✓ Feasible (with small slack)

Q: How many samples are sufficient?

Our Result (informally)

\(O \left( \frac{\log m + \log n}{\varepsilon^2} \right)\) samples are sufficient for testing if players are implementing a Nash equilibrium.

\(\varepsilon\): slack parameter in payoffs
Our Result (informally)

$O\left(\frac{\log m + \log n}{\varepsilon^2}\right)$ samples are sufficient for testing if players are implementing a Nash equilibrium.
Our Result (informally)

\(O \left( \frac{\log m + \log n}{\varepsilon^2} \right)\) samples are sufficient for testing if players are implementing a Nash equilibrium.

**Nash Eq.**: No player can benefit, in expectation, by unilateral deviation.

**\(\varepsilon\)-Nash Eq.**: No player can benefit more than \(\varepsilon\), in expectation, by unilateral deviation.
Our Result (informally)

\[ O \left( \frac{\log m + \log n}{\varepsilon^2} \right) \] samples are sufficient for testing if players are implementing a Nash equilibrium.

Nash Eq.: No player can benefit, in expectation, by unilateral deviation.
\( \varepsilon \)-Nash Eq.: No player can benefit more than \( \varepsilon \), in expectation, by unilateral deviation.

\( s_i \leftarrow \text{empirical distribution over } O \left( \frac{\log m + \log n}{\varepsilon^2} \right) \text{ samples drawn from } x_i \)
Our Result (informally)

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Nash Eq.: No player can benefit, in expectation, by unilateral deviation.
\(\epsilon\)-Nash Eq.: No player can benefit more than \(\epsilon\), in expectation, by unilateral deviation.

\(s_i \leftarrow \) empirical distribution over \( O\left(\frac{\log m + \log n}{\epsilon^2}\right) \) samples drawn from \(x_i\)

Test whether or not the empirical distribution \( \prod_i s_i \) an \(\epsilon\)-Nash equilibrium.
Our Result (informally)

$O\left(\frac{\log m + \log n}{\varepsilon^2}\right)$ samples are sufficient for testing if players are implementing a Nash equilibrium.

$s_i \leftarrow$ empirical distribution over $O\left(\frac{\log m + \log n}{\varepsilon^2}\right)$ samples drawn from $x_i$

If $\prod_i x_i$ is a Nash eq. then $\prod_i s_i$ is an $\varepsilon$-Nash eq., w.h.p.
Our Result (informally)

\[ O \left( \frac{\log m + \log n}{\varepsilon^2} \right) \] samples are sufficient for testing if players are implementing a Nash equilibrium.

\[ s_i \leftarrow \text{empirical distribution over } O \left( \frac{\log m + \log n}{\varepsilon^2} \right) \text{ samples drawn from } x_i \]

If \( \prod_i x_i \) is a Nash eq. then \( \prod_i s_i \) is an \( \varepsilon \)-Nash eq., w.h.p.

If \( \prod_i x_i \) is not a \( 2\varepsilon \)-Nash eq. then \( \prod_i s_i \) is not an \( \varepsilon \)-Nash eq., w.h.p.
Our Result (informally)

\[ O \left( \frac{\log m + \log n}{\varepsilon^2} \right) \] samples are sufficient for testing if players are implementing a Nash equilibrium.

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If \( \prod_i x_i \) is a \( \delta \)-Nash eq. then \( \prod_i s_i \) is an \( (\delta + \varepsilon) \)-Nash eq., w.h.p.

If \( \prod_i x_i \) is not a \( \delta + 2\varepsilon \)-Nash eq. then \( \prod_i s_i \) is not an \( (\delta + \varepsilon) \)-Nash eq., w.h.p.
Our Result (informally)

\[ O \left( \frac{\log m + \log n}{\epsilon^2} \right) \] samples are sufficient for testing if players are implementing a Nash equilibrium.

\[ s_i \leftarrow \text{empirical distribution over} \ O \left( \frac{\log m + \log n}{\epsilon^2} \right) \] samples drawn from \( x_i \)

Main Theorem

If \( \prod_i x_i \) is a Nash eq. then \( \prod_i s_i \) is an \( \epsilon \)-Nash eq., w.h.p.

If \( \prod_i x_i \) is not a \( 2\epsilon \)-Nash eq. then \( \prod_i s_i \) is not an \( \epsilon \)-Nash eq., w.h.p.
Main Theorem

If $\prod_i x_i$ is a Nash eq. then $\prod_i s_i$ is an $\varepsilon$-Nash eq., w.h.p.

If $\prod_i x_i$ is not a $2\varepsilon$-Nash eq. then $\prod_i s_i$ is not an $\varepsilon$-Nash eq., w.h.p.
Proof Sketch

Main Theorem

If $\prod_i x_i$ is a Nash eq. then $\prod_i s_i$ is an $\varepsilon$-Nash eq., w.h.p.

$n$-player $m$-action game

$x_i$ : Mixed strategy of player $i$

$O\left(\frac{\log m + \log n}{\varepsilon^2}\right)$ samples from each $x_i$

$s_i$ : Empirical distribution associated with $x_i$
Proof Sketch

**Main Theorem**

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$u_i : [m]^n \rightarrow [0, 1]$, utility of player $i$
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**Main Theorem**

If $\prod x_i$ is a Nash eq. then $\prod s_i$ is an $\varepsilon$-Nash eq., w.h.p.

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$u_i : [m]^n \rightarrow [0, 1]$, utility of player $i$

$x = \prod x_i$ is a **Nash eq.** if $u_i(a, x_{-i}) \leq u_i(x)$, for all $i, a \in [m]$. 
Proof Sketch

Main Theorem

If $\prod_i x_i$ is a Nash eq. then $\prod_i s_i$ is an $\varepsilon$-Nash eq., w.h.p.

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$u_i : [m]^n \rightarrow [0, 1]$, utility of player $i$

$s = \prod_i s_i$ is an $\varepsilon$-Nash eq. if $u_i(a, s_{-i}) \leq u_i(s) + \varepsilon$, for all $i, a \in [m]$
Proof Sketch

Main Theorem

If $\prod_i x_i$ is a Nash eq. then $\prod_i s_i$ is an $\varepsilon$-Nash eq., w.h.p.

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$u_i : [m]^n \to [0, 1]$, utility of player $i$

To Prove

If $x = \prod_i x_i$ satisfies $u_i(a, x_{-i}) \leq u_i(x)$ then $s = \prod_i s_i$ satisfies, $u_i(a, s_{-i}) \leq u_i(s) + \varepsilon$ w.h.p.
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Key Lemma

Let \( s_i^k \) be the empirical dist. over \( k \) i.i.d. samples from \( x_i \), for all \( i \). Then,

\[
\Pr(\left| u_i(a, s_{-i}^k) - u_i(a, x_{-i}) \right| \geq \varepsilon/2) \leq e^{-c\varepsilon^2 k} \quad a \in [m]
\]
Proof Sketch

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If \( x = \prod_i x_i \) satisfies \( u_i(a, x_i) \leq u_i(x) \) then 
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KeyLemma

Let \( s^k_i \) be the empirical dist. over \( k \) i.i.d. samples from \( x_i \), for all \( i \). Then,

\[
\Pr\left( |u_i(a, s^k_i) - u_i(a, x_i)| \geq \varepsilon/2 \right) \leq e^{-c\varepsilon^2k} \quad a \in [m]
\]

For \( n = 2 \) [LMM03], the concentration result follows from Hoeffding’s inequality.
Proof Sketch

**To Prove**

If \( x = \prod_i x_i \) satisfies \( u_i(a, x_i) \leq u_i(x) \) then \( s = \prod_i s_i \) satisfies, \( u_i(a, s_i) \leq u_i(s) + \varepsilon \) w.h.p.

**Key Lemma**

Let \( s_i^k \) be the empirical dist. over \( k \) i.i.d. samples from \( x_i \), for all \( i \). Then,

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\Pr(\left| u_i(a, s_i^k) - u_i(a, x_i) \right| \geq \varepsilon/2) \leq e^{-c\varepsilon^2 k} \quad a \in [m]
\]

For \( n \geq 3 \), more complicated arguments are required!
Since \( s \) is the product of empirical distributions.
Proof Sketch

**To Prove**

If \( x = \prod_i x_i \) satisfies \( u_i(a, x_{-i}) \leq u_i(x) \) then
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\]

\[
k = O\left( \frac{\log m + \log n}{\varepsilon^2} \right) \quad \Rightarrow \quad e^{-c\varepsilon^2 k} < \frac{1}{nm}
\]
Proof Sketch

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If \( x = \prod_i x_i \) satisfies \( u_i(a, x_{-i}) \leq u_i(x) \) then
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\]

\[ k = O \left( \frac{\log m + \log n}{\varepsilon^2} \right) \Rightarrow \text{Ineqs. hold w.h.p.} \]
\[ \forall i \in [n], \forall a \in [m] \]
Proof Sketch

To Prove

If \( x = \prod_i x_i \) satisfies \( u_i(a, x_{-i}) \leq u_i(x) \) then
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Let \( s_i^k \) be the empirical dist. over \( k \) i.i.d. samples from \( x_i \), for all \( i \). Then,

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\Pr(|u_i(a, s_{-i}^k) - u_i(a, x_{-i})| \geq \varepsilon/2) \leq e^{-c\varepsilon^2 k} \quad a \in [m]
\]

\[
u_i(a, s_{-i}) < u_i(a, x_{-i}) + \frac{\varepsilon}{2} \leq u_i(s_i, x_{-i}) + \frac{\varepsilon}{2} < u_i(s_i, s_{-i}) + \varepsilon
\]
Main Theorem ✓

If $\prod_i x_i$ is a Nash eq. then $\prod_i s_i$ is an $\epsilon$-Nash eq., w.h.p.

$O\left(\frac{\log m + \log n}{\epsilon^2}\right)$ samples from each $x_i$

$s_i :$ Empirical distribution associated with $x_i$

Lower Bounds

$\Omega(\log m)$ samples are necessary [Althöfer ’94]

$\Omega(\log n)$ samples are necessary (2n players playing matching pennies in pairs)
Main Theorem

If $\prod_i x_i$ is a Nash eq. then $\prod_i s_i$ is an $\varepsilon$-Nash eq., w.h.p.

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Small-Support Approximate Equilibrium

Every $n$-player $m$-action game admits a small-support $\varepsilon$-Nash eq.
Main Theorem

If $\prod_i x_i$ is a Nash eq. then $\prod_i s_i$ is an $\varepsilon$-Nash eq., w.h.p.

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Small-Support Approximate Equilibrium

Every $n$-player $m$-action game admits an $\varepsilon$-Nash eq. in which the strategy of every player is a uniform distribution of support size at most $O \left( \frac{\log m + \log n}{\varepsilon^2} \right)$. 
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Every $n$-player $m$-action game admits an $\varepsilon$-Nash eq. in which the strategy of each player is a uniform distribution of support size at most $O\left(\frac{\log m + \log n}{\varepsilon^2}\right)$.

Improves upon previously known support-size bounds

- $O\left(\frac{n^2 \log m}{\varepsilon^2}\right)$ [LMM03]
- $O\left(\frac{n \log m}{\varepsilon^2}\right)$ [HRS08]
Main Theorem

If $\prod_i x_i$ is a Nash eq. then $\prod_i s_i$ is an $\varepsilon$-Nash eq., w.h.p.

Small-Support Approximate Equilibrium

Every $n$-player $m$-action game admits an $\varepsilon$-Nash eq. in which the strategy of each player is a uniform distribution of support size at most $O \left( \frac{\log m + \log n}{\varepsilon^2} \right)$.

Implies a $\text{poly} \left( m^n \left( \frac{\log m + \log n}{\varepsilon^2} \right) \right)$-time algorithm for determining $\varepsilon$-Nash eq.

Improves upon previous known computational bounds [LMM03, Nisan09, DP09]; in particular, for large number of players and constant number of actions per player: $N^{\log \log \log N}$
Test if players are implementing a **Nash equilibrium** using i.i.d samples in large games ✓

Test if players are implementing a **correlated equilibrium** using i.i.d samples in large games

Test if players are implementing a **coarse correlated equilibrium** using i.i.d samples in large games
### Definition

Distribution $x$ is said to be a **correlated equilibrium (CE)** if for all $i$ and for all (switching rules) $f : [m] \rightarrow [m]$

$$
\mathbb{E}_{a \sim x} [u_i(f(a_i), a_{-i})] \leq \mathbb{E}_{a \sim x} [u_i(a)]
$$

### Definition

Distribution $y$ is said to be a **$\varepsilon$ correlated equilibrium ($\varepsilon$-CE)** if for all $i$ and for all (switching rules) $f : [m] \rightarrow [m]$

$$
\mathbb{E}_{a \sim y} [u_i(f(a_i), a_{-i})] \leq \mathbb{E}_{a \sim y} [u_i(a)] + \varepsilon
$$

$x$ and $s$ might not be product distributions.
**Definition**

Distribution $x$ is said to be a **correlated equilibrium** (CE) if for all $i$ and for all (switching rules) $f : [m] \rightarrow [m]$

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\mathbb{E}_{a \sim x} [u_i(f(a_i), a_{-i})] \leq \mathbb{E}_{a \sim x} [u_i(a)]
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**Definition**

Distribution $y$ is said to be a **$\varepsilon$ correlated equilibrium** ($\varepsilon$-CE) if for all $i$ and for all (switching rules) $f : [m] \rightarrow [m]$

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\mathbb{E}_{a \sim y} [u_i(f(a_i), a_{-i})] \leq \mathbb{E}_{a \sim y} [u_i(a)] + \varepsilon
$$

**Theorem**

If $x$ is a CE then the empirical distribution over $O\left(\frac{m \log m + \log n}{\varepsilon^2}\right)$ samples forms a $\varepsilon$-CE w.h.p.
### Testing Results

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Thank you!