3 Probability

- 3.1 Basic Ideas
- 3.2 Conditional Probability and Independence
- 3.3 Random Variables
- 3.4 Functions of Random Variables

### 3.1 Basic Ideas

A *sample space*, $S$, is the set of ________________ of a random process.

An *event*, $A$, is a ____________ of $S$.

e.g. Consider tossing two fair coins.

- The sample space is $S =$ ________________ (draw tree)
- Events include
  - $A =$ both heads = ________________
  - $B =$ at least one head = ________________
  - $C =$ three heads = ________________

**Combining Events**

Set notation is convenient for describing compound events:

- $A \cup B =$ “$A$ _____ $B$” = union of $A$ and $B$ = \{ outcomes in ________________ \}
- $A \cap B =$ “$A$ _____ $B$” = intersection of $A$ and $B$ = \{ outcomes in ________________ \}
- $A^c =$ “_____ $A$” = complement of $A$ = \{ outcomes that ________________ to $A$ \}

e.g. For tossing two coins,

- $A \cup B =$ ________________
- $A \cap B =$ ________________
- $A^c =$ ________________
- $A \cup A^c =$ ________________
- $A \cap A^c =$ ________________
- $A^c \cup B =$ ________________
- $A^c \cap B =$ ________________
Mutually Exclusive Events

Events $A$ and $B$ are *mutually exclusive* if they have \( A \cap B = \emptyset \). Events $A_1, \cdots, A_n$ are mutually exclusive if \( \bigcap_{i=1}^{n} A_i \) has no outcomes in common. 

E.g. For tossing two coins, __________________ are mutually exclusive.

Axioms of Probability

The *probability* of an outcome of a random process is the \( \ldots \) of times the outcome would occur \( \ldots \), if the process were to be repeated \( \ldots \).

E.g. Here are results of computer simulation of $n$ random coin tosses:

<table>
<thead>
<tr>
<th>$n$</th>
<th>Data</th>
<th>#Heads</th>
<th>#Tails</th>
<th>$P(\text{Heads}) \approx \frac{\text{#Heads}}{n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>T</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>TTTHTHTHTHT</td>
<td>4</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>HHTHTHTHTHTHT</td>
<td>53</td>
<td>47</td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>HHTHTHTHTHTHT</td>
<td>491</td>
<td>509</td>
<td></td>
</tr>
<tr>
<td>1000000000</td>
<td>THHTTHHHHHT</td>
<td>500002628</td>
<td>499997372</td>
<td></td>
</tr>
</tbody>
</table>

As \( \ldots \), the proportion of heads is approaching the long-run proportion \( \ldots \), which is $P(\text{Heads})$.

The probability of an event $A$ is denoted $P(A)$.

Axioms of probability include

- $P(S) = \ldots$

- For any event $A$, \( \ldots \)

- For mutually exclusive events $A$ and $B$, $P(A \cup B) = \ldots$; for mutually exclusive $A_1, A_2, \cdots$, $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$

Consequences include

- For any event $A$, $P(A^c) = \ldots$

- $P(\emptyset) = \ldots$
e.g. Here is the distribution of Canadian responses to the question, “What is your mother tongue?”

<table>
<thead>
<tr>
<th>Language</th>
<th>English</th>
<th>French</th>
<th>Asian/Pacific</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.59</td>
<td>0.23</td>
<td>0.07</td>
<td>x</td>
</tr>
</tbody>
</table>

- \( x = \)
- \( P(\text{mother tongue isn’t English}) = \)
- \( P(\text{mother tongue is English or French}) = \)

e.g. For tossing two coins, each of the four outcomes is equally likely, so

- \( P(\text{HH}) = P(\text{HT}) = P(\text{TH}) = P(\text{TT}) = \) __________
- \( P(\text{A}) = \) __________
- \( P(\text{B}) = \) __________
- \( P(\text{B}^c) = \) __________
- \( P(\text{C}) = \) __________

**Sample Spaces with Equally Likely Outcomes**

If \( S \) is a sample space with \( N \) equally-likely outcomes, then the probability of each outcome is __________. If \( A \) is an event containing \( k \) outcomes, then \( P(A) = \) __________.

e.g. Consider drawing a card randomly from a 52-card deck. \( P(\text{Ace of spades}) = \) __________, \( P(\text{Ace}) = \) __________, and \( P(\text{spade}) = \) __________.

**The Addition Rule**

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \] (draw overlapping circles)

- e.g. \( P(\text{Ace or spade}) = \)

Note that the axiom “for mutually exclusive \( A \) and \( B \), \( P(A \cup B) = P(A) + P(B) \)” is the special case where __________.
Examples

e.g. (p. 73 #3) Of a certain manufacturer’s silicon wafers, 10% have resistances below specification and 5% have resistances above specification.

- What is the probability that the resistance of a randomly chosen wafer does not meet the specification?

- If a randomly chosen wafer has a resistance that does not meet the specification, what is the probability that it’s too low? (Hint: draw a picture.)

e.g. (#6) Human blood may contain either or both of two antigens, A and B. Type A blood contains only antigen A, type B blood contains only antigen B, type AB blood contains both antigens, and type O blood contains neither. At a certain blood bank, 35% of donors have type A blood, 10% have type B, and 5% have type AB.

- What is the probability that a randomly chosen donor is type O?

- A type A recipient may receive blood from a donor whose blood doesn’t contain the B antigen. What is the probability that a randomly chosen donor may donate to a type A recipient?

e.g. (#7) 60% of purchases at a computer store are desktops, 30% are laptops, and 10% are printers. An audit samples one purchase record at random.

- What is the probability that it’s a desktop?

- What is the probability that it is a desktop or laptop?