3.2 Conditional Probability and Independence

Conditional Probability

The conditional probability of event $A$, given that event $B$ occurred, is denoted “__________” (and pronounced “probability of $A$, given $B$”).

e.g. Consider the sample space $S = \{ $ outcomes of two fair die rolls $\} $ and these events:

- $A = $ first die is a 3
- $B = $ second die is a 1
- $C = $ the dice sum to 8

Then $|S| =$ ___________ and

- $P(A) =$ ___________
- $P(B) =$ ___________
- $P(C) =$ ___________
- $P(A \cap B) =$ ___________
- $P(A \cap C) =$ ___________
- $P(B \cap C) =$ ___________

Now suppose we know $A$ occurred. Then

- $P(B|A) =$ ___________
- $P(C|A) =$ ___________

If, instead, we know $B$ occurred, then

- $P(A|B) =$ ___________
- $P(C|B) =$ ___________

These quantities are related by the conditional probability formula $P(A|B) = \frac{P(A \cap B)}{P(B)}$ (draw).

Rearranging gives $P(A \cap B) =$ ___________ (the book’s “Multiplication Rule”).

(Or, swapping $A$ and $B$, the original conditional probability formula becomes $P(B|A) =$ ___________ so that $P(A \cap B) =$ _________________.)
Independent Events

Two events $A$ and $B$ are independent if and only if $P(A \cap B) =$ __________. A collection of events is mutually independent if and only if, for every finite subset $A_1, \ldots, A_n$, we have $P(A_1 \cap \cdots \cap A_n) = P(A_1) \cdots P(A_n)$.

e.g. Regarding rolling two dice, events __________ and __________ are independent.

For independent $A$ and $B$, the conditional probability formula, $P(A|B) = \frac{P(A \cap B)}{P(B)}$, implies $P(A|B) =$ ________________ and $P(B|A) =$ ________________.

(Note: The book swaps the definition and consequence.)

e.g. (p. 83 #5) A geneticist is studying two genes. Each gene can be either dominant or recessive. A sample of 100 individuals is categorized as follows:

<table>
<thead>
<tr>
<th>Gene 1</th>
<th>Gene 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Dominant</td>
</tr>
<tr>
<td>Dominant</td>
<td>56</td>
</tr>
<tr>
<td>Recessive</td>
<td>14</td>
</tr>
</tbody>
</table>

For a randomly sampled individual, find

a. $P$(Gene 1 is dominant) =

b. $P$(Gene 2 is dominant) =

c. $P$(Gene 2 is dominant | Gene 1 is dominant) =

d. These genes are in “linkage equilibrium” if the events “Gene 1 is dominant” and “Gene 2 is dominant” are independent. Are they in equilibrium?

A Few Set Algebra Formulas (draw)

- $(A \cup B)^c = A^c \cap B^c$
\[(A \cap B)^c = A^c \cup B^c\]

\[A \cup (B \cap C) = (A \cup B) \cap (A \cup C)\]

\[A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\]

**Application to Reliability Analysis**

- If a system relies on *both* of two independent parts A and B, then \(P(\text{system success}) = \) ________________. This corresponds to wiring ________________. (draw)

- If a system relies on *either* of two independent parts A and B, then \(P(\text{system success}) = \) ________________. This corresponds to wiring ________________. (draw)

e.g. Suppose you plan to sleep 1500 feet up a stone, and you have two independent carabiners, A and B, each of which has a 90% chance of holding for the night.

- If you hang from them in series, then \(P(\text{you live}) = \)

- If you hang from them in parallel, then \(P(\text{you live}) = \)
e.g. Given independent components $A, B, C, D,$ and $E,$ with failure probabilities $1/2, 1/3, 1/4, 1/5,$ and $1/6,$ find the probability of success of a system that connects $A, B,$ and $C$ in parallel, $D$ and $E$ in series, and the two subsystems in parallel. (Hint: draw a picture.)

More Examples (if time allows)

e.g. (Part of p. 84 # 10) Show that if $A$ and $B$ are independent, then $A^c$ and $B^c$ are independent.