7.2 Inferences on the Difference Between Two Proportions

We compare two population proportions, \( p_X \) and \( p_Y \), by studying their difference, \( p_X - p_Y \).

Notation:

<table>
<thead>
<tr>
<th>Success probability</th>
<th>Population 1</th>
<th>Population 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>#Trials</td>
<td>( n_X )</td>
<td>( n_Y )</td>
</tr>
<tr>
<td>#Successes</td>
<td>( X )</td>
<td>( Y )</td>
</tr>
<tr>
<td>Sample proportion of successes</td>
<td>( \hat{p}_X = \frac{X}{n_X} )</td>
<td>( \hat{p}_Y = \frac{Y}{n_Y} )</td>
</tr>
</tbody>
</table>

For inference about \( p_X - p_Y \), use the statistic \( \hat{p}_X - \hat{p}_Y \), and then

- find a confidence interval for \( p_X - p_Y \)
- test \( H_0 : p_X - p_Y = 0 \) (\( \Rightarrow p_X = p_Y \))

To do this, we need the distribution of \( \hat{p}_X - \hat{p}_Y \). Recall for independent \( X \) and \( Y \):

- If \( X \sim N(\mu_X, \sigma_X^2) \) and \( Y \sim N(\mu_Y, \sigma_Y^2) \), then \( X - Y \sim \)

- If \( X \sim \text{Bin}(n, p) \), and \( np > 10 \) and \( n(1 - p) > 10 \), then \( X \sim N(\text{________, \text{________}}) \) (\( \approx \); because CLT applies to \( X = \sum_{i=1}^n B_i \), where \( B_i \sim \text{Bernoulli}(p) \) (\( \S 4.8 \))

\[ \Rightarrow \hat{p} = \frac{X}{n} \sim \]

It follows that, for \( n_X p_X > 10, n_X (1 - p_X) > 10, n_Y p_Y > 10, \) and \( n_Y (1 - p_Y) > 10 \),

\[ \hat{p}_X - \hat{p}_Y \sim \]

We need the standard deviation for inference about the unknown \( p_X - p_Y \), but we don’t know \( \text{________} \) or \( \text{________} \). If the #successes and #failures are more than \( \text{________} \)in each sample, we can approximate them with \( \text{________} \) and \( \text{________} \).

Confidence Intervals on the Difference Between Two Proportions

Recall that many confidence intervals have the form

\[
\text{(point estimate)} \pm \text{(margin of error)}
\]

\[= \text{(point estimate)} \pm \text{(table value for confidence)} \times \text{[estimated or true standard deviation of point estimate] } \]

\[= \hat{\theta} \pm \text{(table value for confidence)} \times \sigma_{\hat{\theta}} \]
The Old Confidence Interval

If the successes and failures are more than 10 in each sample, then the old 100%(1−α) confidence interval for \( p_X - p_Y \) is

\[
(\hat{p}_X - \hat{p}_Y) \pm z_{\alpha/2} \sqrt{\frac{\hat{p}_X(1-\hat{p}_X)}{n_X} + \frac{\hat{p}_Y(1-\hat{p}_Y)}{n_Y}}
\]

For small samples, this interval _______________ \( p_X - p_Y \) for a proportion 1 − α of samples.

The New Plus-Four Confidence Interval

Recent research (2000) describes an improvement: add four fake observations, two successes and two failures, _______________ to each sample. (The §5.3 plus-four interval for a single proportion added _____ successes and _____ failures to the single sample.)

Let independent \( X \sim \text{Bin}(n_X, p_X) \) and \( Y \sim \text{Bin}(n_Y, p_Y) \). Define

\[
\tilde{n}_X = \_, \tilde{n}_Y = \_, \tilde{p}_X = \_, \text{ and } \tilde{p}_Y = \_
\]

Then the (100%)(1−α) plus-four confidence interval for \( p_X - p_Y \) is

\[
(\tilde{p}_X - \tilde{p}_Y) \pm z_{\alpha/2} \sqrt{\frac{\tilde{p}_X(1-\tilde{p}_X)}{\tilde{n}_X} + \frac{\tilde{p}_Y(1-\tilde{p}_Y)}{\tilde{n}_Y}}
\]

This interval can be used if \( n_X > 4 \) and \( n_Y > 4 \), without regard for the successes and failures. (Since \( (p_X - p_Y) \in [\_, \_] \), trim the interval if it extends outside \([\_, \_]\).)

e.g. A randomized double-blind experiment assigned 244 smokers who wanted to quit to receive nicotine patches and another 245 to receive patches and an antidepressant. After a year, 40 in the first group and 87 in the second had quit. Give a 99% plus-four confidence interval for the difference (treatment − control) in proportions of smokers who quit.
Hypothesis Tests on the Difference Between Two Proportions

Many hypothesis tests (including those from §6.1, 6.3, 6.4, and 7.1) use test statistics of the form

\[
\frac{\text{(point estimate)} - \text{(parameter value under } H_0)}{\text{(estimated or true) standard deviation of point estimate}}
\]

This point estimate tells how far the estimate is from the parameter, in units of the standard deviation of the point estimate.

Our point estimate is \( \hat{p}_X - \hat{p}_Y \), which has standard deviation \( \sqrt{\frac{\hat{p}_X(1-\hat{p}_X)}{n_X} + \frac{\hat{p}_Y(1-\hat{p}_Y)}{n_Y}} \).

Under \( H_0 : p_X - p_Y = 0 \), it’s best not to use \( p_X \approx \hat{p}_X \) and \( p_Y \approx \hat{p}_Y \) because \( H_0 \Rightarrow \) \( \text{something} \). Instead, estimate both with the proportion

\[
\hat{p} = \frac{\# \text{successes in both samples combined}}{\text{combined sample size}} = \text{something}
\]

Then use \( \hat{p} \) for both \( \text{something} \) and \( \text{something} \) in the expression for the standard deviation.

Here is the test for a difference of two proportions:

Let independent \( X \sim \text{Bin}(n_X, p_X) \) and \( Y \sim \text{Bin}(n_Y, p_Y) \), with \( n_X p_X, n_X (1-p_X), n_Y p_Y, \) and \( n_Y (1-p_Y) \) all > 10. To test \( H_0 : p_X - p_Y = 0 \):

1. State null and alternative hypotheses, \( H_0 \) and \( H_1 \)
2. Check assumptions
3. Find \( \hat{p}_X = \frac{X}{n_X}, \hat{p}_Y = \frac{Y}{n_Y} \), and pooled \( \hat{p} = \frac{X + Y}{n_X + n_Y} \)
4. Find the test statistic, \( z = \frac{(\hat{p}_X - \hat{p}_Y) - 0}{\sqrt{\hat{p}(1-\hat{p})(1/n_X + 1/n_Y)}} \)
5. Find the \( P \)-value, which is an area under the \( N(0,1) \) curve depending on \( H_1 \):
   - \( H_1 : p_X - p_Y > 0 \) \( \Rightarrow \) \( P \)-value = \( P(Z > z) \), the area right of \( z \)
   - \( H_1 : p_X - p_Y < 0 \) \( \Rightarrow \) \( P \)-value = \( P(Z < z) \), the area left of \( z \)
   - \( H_1 : p_X - p_Y \neq 0 \) \( \Rightarrow \) \( P \)-value = \( P(|Z| > |z|) \), the sum of the two tail areas
6. Draw a conclusion
e.g. Referring to the previous example, how significant is the evidence that the antidepressant increases the success rate?

e.g. In the early 1950s, a randomly-selected 200745 children were given Jonas Salk’s new vaccine for polio, and another 201229 were given a placebo. 82 in the vaccine group got polio, while 162 in the placebo group got polio. Test whether the vaccine was effective.