9.3 Two-Factor Experiments (part 2 of 2)

Assumptions

1. The design is ____________
2. The design is ____________
3. $K \geq ____$
4. Within each treatment $ij$, the observations $X_{ij1}, \ldots, X_{ijK}$ are a ________________ from a ____________ population
5. All treatment populations have ________________

Checks include:

- A residual plot of residuals $X_{ijk} - \bar{X}_{ij}$ vs. fitted values $\bar{X}_{ij}$, made by plotting the points $\{(\bar{X}_{ij}, X_{ijk} - \bar{X}_{ij})\}$ checks for equal variances: points should show no ________________ and no ________________
- A ____________ probability plot of residuals $X_{ijk} - \bar{X}_{ij}$ should be $\approx$ ________________

Test Statistics for Two-Way ANOVA

The test statistics require ____________ squares from ____________ squares from ________________.

Sample Means

- ____________ mean $\bar{X}_{ij} = \frac{1}{K} \sum_{k=1}^{K} X_{ijk}$
- ____________ mean $\bar{X}_{i.} = \frac{1}{J} \sum_{j=1}^{J} \bar{X}_{ij} = \frac{1}{JK} \sum_{j=1}^{J} \sum_{k=1}^{K} X_{ijk}$
- ____________ mean $\bar{X}_{.j} = \frac{1}{I} \sum_{i=1}^{I} \bar{X}_{ij} = \frac{1}{IK} \sum_{i=1}^{I} \sum_{k=1}^{K} X_{ijk}$
- ____________ mean $\bar{X}_{..} = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \bar{X}_{ij}$ (average of observations)

\[
[ = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \bar{X}_{ij} \text{ (average of cell means)}
\]
\[
= \frac{1}{J} \sum_{i=1}^{I} \bar{X}_{i.} \text{ (average of row means)}
\]
\[
= \frac{1}{I} \sum_{j=1}^{J} \bar{X}_{.j} \text{ (average of column means)}
\]
e.g. Find sample means for the data (p. 437 #6) on tool lifetime vs. feed rate and speed. Estimate
the main effects $\{\alpha_i\}$ and $\{\beta_j\}$ and the interactions $\{\gamma_{ij}\}$.

<table>
<thead>
<tr>
<th>Feed Rate</th>
<th>Speed</th>
<th>Medium</th>
<th>Fast</th>
<th>Row Means</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light</td>
<td>Slow</td>
<td>$X_{11} = 59.675$</td>
<td>$X_{12} = 59.550$</td>
<td>$X_{13} = 55.200$</td>
</tr>
<tr>
<td></td>
<td>$\hat{\gamma}_{11} = -1.22$</td>
<td>$\hat{\gamma}_{12} = .87$</td>
<td>$\hat{\gamma}_{13} = .35$</td>
<td></td>
</tr>
<tr>
<td>Medium</td>
<td>$X_{21} = 51.2$</td>
<td>$X_{22} = 49.650$</td>
<td>$X_{23} = 47.125$</td>
<td>$\bar{X}_{2..} = _________$</td>
</tr>
<tr>
<td></td>
<td>$\hat{\gamma}_{21} = ________$</td>
<td>$\hat{\gamma}_{22} = -1.16$</td>
<td>$\hat{\gamma}_{23} = 1.14$</td>
<td></td>
</tr>
<tr>
<td>Heavy</td>
<td>$X_{31} = 44.8$</td>
<td>$X_{32} = 41.050$</td>
<td>$X_{33} = 36.450$</td>
<td>$\bar{X}_{3..} = 41.225$</td>
</tr>
<tr>
<td></td>
<td>$\hat{\gamma}_{31} = 2.20$</td>
<td>$\hat{\gamma}_{32} = -0.71$</td>
<td>$\hat{\gamma}_{33} = -1.49$</td>
<td></td>
</tr>
</tbody>
</table>

Column Means

$\bar{X}_{1..} = \_\_\_\_\_\_\_\_$ $\bar{X}_{2..} = 50.083$ $\bar{X}_{3..} = 46.258$ $\bar{X}_{...} = \_\_\_\_\_\_\_\_$

Notes:

- $I = \_\_\_\_\_\_\_$, $J = \_\_\_\_\_\_\_\_$, $K = \_\_\_\_\_\_\_\_\_$
- $\hat{X}_{21} = \_\_\_\_\_\_\_\_\_$, $\hat{X}_{2} = \_\_\_\_\_\_\_\_\_\_$
- $\hat{X}_{...} = \_\_\_\_\_\_\_\_\_\_$, $\hat{\alpha}_{2} = \_\_\_\_\_\_\_\_\_\_$
- $\hat{X}_{.1} = \_\_\_\_\_\_\_\_\_$, $\hat{\beta}_{1} = \_\_\_\_\_\_\_\_\_\_$
- $\hat{\gamma}_{21} = \_\_\_\_\_\_\_\_\_\_$

Sums of Squares

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>Sum of Squares</th>
<th>Example DF SS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows (SSA)</td>
<td>$I - 1$</td>
<td>$JK \sum \hat{\alpha}<em>{i}^2 = JK \sum</em>{i=1}^{I} (\bar{X}<em>{i..} - \bar{X}</em>{...})^2$</td>
<td>2 1718</td>
</tr>
<tr>
<td>Columns (SSB)</td>
<td>$J - 1$</td>
<td>$IK \sum \hat{\beta}<em>{j}^2 = IK \sum</em>{j=1}^{J} (\bar{X}<em>{.j..} - \bar{X}</em>{...})^2$</td>
<td>_ _ _ _ _ _ _ _</td>
</tr>
<tr>
<td>Interactions (SSAB)</td>
<td>$(I - 1)(J - 1)$</td>
<td>$K \sum \sum \hat{\gamma}<em>{ij}^2 = K \sum</em>{i=1}^{I} \sum_{j=1}^{J} (\bar{X}<em>{ij..} - (\bar{X}</em>{...} + \hat{\alpha}<em>{i} + \hat{\beta}</em>{j}))^2$</td>
<td>4 48.8</td>
</tr>
<tr>
<td>Error (SSE)</td>
<td>$IJ(K - 1)$</td>
<td>$\sum \sum \sum \text{residual}^2_{ijk} = \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (X_{ijk} - \bar{X}_{ij..})^2$</td>
<td>$27$ 197.5</td>
</tr>
<tr>
<td>Total (SST)</td>
<td>$IJK - 1$</td>
<td>$\sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{k=1}^{K} (X_{ijk} - \bar{X}_{...})^2$</td>
<td>35 2189</td>
</tr>
</tbody>
</table>
Mean Squares

Divide each sum of squares (SS) by \(\frac{I-1}{I-1}\) to get a corresponding mean square (MS):

<table>
<thead>
<tr>
<th>Mean Square</th>
<th>Use</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSA = (\frac{SSA}{I-1})</td>
<td>MSA (\Rightarrow) reject (H_0 : \alpha_1 = \cdots = \alpha_I = 0)</td>
<td>859.2</td>
</tr>
<tr>
<td>MSB = (\frac{SSB}{J-1})</td>
<td>MSB (\Rightarrow) reject (H_0 : \beta_1 = \cdots = \beta_J = 0)</td>
<td>---</td>
</tr>
<tr>
<td>MSAB = (\frac{SSAB}{(I-1)(J-1)})</td>
<td>MSAB (\Rightarrow) reject (H_0 : \gamma_1 = \cdots = \gamma_{IJ} = 0)</td>
<td>12.2</td>
</tr>
</tbody>
</table>

MSE is the mean square ______. It depends on distances between ______ and their ______ means, but not on row or column effects or interactions. It’s an estimate of ______, measuring random variation inherent in the process.

MSE = \(\frac{SSE}{IJ(K-1)}\)

Test Statistics

The test statistics are ratios of MSA, MSB, and MSAB to ______. Each ratio has an ______ distribution under its respective null hypothesis:

- \(H_0 : \alpha_1 = \cdots = \alpha_I = 0 \Rightarrow F = \frac{MSA}{MSE} \sim F_{I-1,IJ(K-1)}\)

- \(H_0 : \beta_1 = \cdots = \beta_J = 0 \Rightarrow F = \frac{MSE}{\text{______}} \sim F_{______,______}\)

- \(H_0 : \gamma_1 = \cdots = \gamma_{IJ} = 0 \Rightarrow F = \frac{MSAB}{MSE} \sim F_{(I-1)(J-1),IJ(K-1)}\)

In each case, the \(P\)-value is a ______ probability from the appropriate \(F\) distribution.
e.g. Answer the questions, “Does feed rate influence lifetime?” and “Does speed influence lifetime?”.

- Does the \( \gamma_{ij} \) model hold? That is, is \( \gamma_{ij} = 0 \) for all \( i \) and \( j \) (so that the model \( \mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij} \) becomes \( \mu_{ij} = \mu \)?)

\[ H_0 : \gamma_{11} = \ldots = \gamma_{IJ} = 0 \]

\( F = \)

\( P\)-value =

Conclusion:

- If the additive model holds,
  - does feed rate influence lifetime?
    \[ H_0 : \alpha_1 = \ldots = \alpha_I = 0 \]
    \( F = \)
    \( P\)-value =

Conclusion:

- does speed influence lifetime?
  \[ H_0 : \beta_1 = \ldots = \beta_J = 0 \]
  \( F = \)
  \( P\)-value =

Conclusion:

The preceding work can be summarized in this two-way ANOVA table:

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rows (SSA)</td>
<td>2</td>
<td>1718.4</td>
<td>859.2</td>
<td>117.7</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>Columns (SSB)</td>
<td>2</td>
<td>224.2</td>
<td>112.1</td>
<td>15.3</td>
<td>&lt;.001</td>
</tr>
<tr>
<td>Interactions (SSAB)</td>
<td>4</td>
<td>48.8</td>
<td>12.2</td>
<td>1.67</td>
<td>&gt;.100</td>
</tr>
<tr>
<td>Error (SSE)</td>
<td>27</td>
<td>197.5</td>
<td>7.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total (SST)</td>
<td>35</td>
<td>2188.8</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Two-Way ANOVA when \( K = 1 \)**

\( K = 1 \implies \text{SSE} = \) \( \implies \text{can't estimate within-treatment variation; \text{MSE} = \frac{\text{SSE}}{IJ(K-1)} = \ldots = \ldots \implies \text{no} \) \text{for row and column effects.} \)

But, if we assume the \( \ldots \) holds \( \ldots \), we can use \( \ldots \) for SSE, and \( \ldots \) for MSE, and proceed with \( \ldots \) for row and column effects.