9.5 Preface

Ronald Fisher (1890-1962), “a __________ who almost single-handedly created the foundations for modern statistical science”, said,

No aphorism is more frequently repeated in connection with field trials, than that we must ask __________ few questions, or, ideally, one question, at a time. The writer is convinced that this view is __________. Nature, he suggests, will best respond to a __________ and __________ questionnaire . . .

A factorial experiment allows estimation of the effect of __________ factors, and _______ __________, with the same _______ a simple experiment requires to estimate any one effect with _______ degree of accuracy.

e.g. (Harold Hotelling, 1895-1973) A pan balance used with a set of standard masses indicates the __________ masses in its two pans. Suppose it has independent random errors with mean 0 and standard deviation $\sigma$. How should we find the masses of four diamonds whose (unknown) true masses are $\theta_1, \theta_2, \theta_3, \text{ and } \theta_4$?

• __________: use the balance _______ times, with diamonds arranged in the pans as follows:

<table>
<thead>
<tr>
<th>Measurement</th>
<th>Left</th>
<th>Right</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>1</td>
<td>empty</td>
<td></td>
</tr>
<tr>
<td>$M_2$</td>
<td>2</td>
<td>empty</td>
<td></td>
</tr>
<tr>
<td>$M_3$</td>
<td>3</td>
<td>empty</td>
<td></td>
</tr>
<tr>
<td>$M_4$</td>
<td>4</td>
<td>empty</td>
<td></td>
</tr>
</tbody>
</table>

For $i = 1 \text{ to } 4$, $\hat{\theta}_i = _______ \text{ has standard deviation } _______ 

(Improving all four $\hat{\theta}_i$'s standard deviations to $\sigma/2 \text{ requires } _______ \text{ measurements.})

• __________: use the balance _______ times, arranging diamonds in pans as follows:

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<th>Measurement</th>
<th>Left</th>
<th>Right</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_3$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$M_4$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The estimates are then

$\hat{\theta}_1 = _______ $

$\hat{\theta}_2 = _______ $

$\hat{\theta}_3 = \frac{1}{4}(M_1 - M_2 + M_3 - M_4)$

$\hat{\theta}_4 = \frac{1}{4}(M_1 - M_2 - M_3 + M_4)$

$\hat{\theta}_1 \text{ has variance } \sigma^2_{\hat{\theta}_1} = \frac{\sigma^2}{4}(M_1 + M_2 + M_3 + M_4) = _______ $

and standard deviation _______.

($\hat{\theta}_2, \hat{\theta}_3, \text{ and } \hat{\theta}_4 \text{ also have standard deviation } _______,)$
9.5 $2^p$ Factorial Experiments (part 1 of 2)

To study $p$ factors simultaneously, a preliminary experiment can be done in which each factor has only ______ levels, ______ and ______. The experiment then has ______ treatments and is called a $2^p$ factorial experiment. It prepares for further experimentation on the important factors at (possibly) _______________ levels per factor.

We’ve already discussed $2^p$ factorial experiments for $p =$ ______ and ______.

e.g. A $2^3$ factorial experiment has ______ factors and ______ treatments.

$2^3$ Factorial Experiments

Consider a $2^p$ factorial experiment with $p = 3$ factors ______, ______, and ______.

Treatments

The treatment with all factors at their ________ levels is denoted “______”. Other treatments are denoted by lower-case character strings, where “______”, “______”, or “______” indicates that the corresponding factor is at its ________ level.

e.g. The treatment

- “1” has all 3 factors __________

- “a” has ____________________ high and ____________________ low

- “ac” has ____________________ high, and ____________________ low

e.g. The 8 treatments are __________________________
The 8 treatments may be pictured as:

```
c --- bc
ac --- abc
1 --- b
```

Corresponding treatment cell means are:

```
X_c --- X_bc
X_ac --- X_abc
X_1 --- X_b
X_a --- X_ab
```

Main Effects

A _______ is a linear combination of treatment means whose coefficients add to _______.
The contrast of a factor is the _______ of mean responses at the factor’s _______ level minus
the _______ of mean responses at the factor’s _______ level.

e.g. The contrast for factor A is

\[
\frac{1}{4}(\bar{X}_a + \bar{X}_{ab} + \bar{X}_{ac} + \bar{X}_{abc}) - \frac{1}{4}(\bar{X}_1 + \bar{X}_b + \bar{X}_c + \bar{X}_{bc})
\]

The main effect of a factor is the difference in its _______ response with the factor at its
_________ level and its _______ response with the factor _______. A main effect estimate is therefore _______ of its contrast.

e.g. The main effect for factor A is denoted \( A \) and estimated as

\[
\frac{1}{4}(\bar{X}_a + \bar{X}_{ab} + \bar{X}_{ac} + \bar{X}_{abc}) - \frac{1}{4}(\bar{X}_1 + \bar{X}_b + \bar{X}_c + \bar{X}_{bc})
\]

Here are the main effects pictured:

- \( A = \) _______ mean minus _______ mean:

```
c --- bc
ac --- abc
1 --- b
```

- \( B = \) _______ mean minus _______ mean:

```
c --- bc
ac --- abc
1 --- b
```
\[ C = \text{mean minus mean:} \]

\[ a \rightarrow b \rightarrow c \]

Interactions

\[ AB = \text{half the difference in the mean } \text{effect with } \text{and the mean } \text{effect with} \]

\[ AC = \text{the diagonal plane } 1, b, ac, abc \text{ mean minus the diagonal plane } a, ab, c, bc \text{ mean:} \]

\[ BC = \text{the diagonal plane } 1, a, bc, abc \text{ mean minus the diagonal plane } b, ab, c, ac \text{ mean:} \]

(The “half” makes the \___________ the same for all main effect and interaction estimates.)
ABC, the one three-way interaction, is half the difference in the mean AB interaction with C high and the mean AB interaction with C low = mean of a, b, c, abc minus mean of 1, ab, ac, bc:

![Diagram of interactions]

Example

e.g. (p. 463 #4) A study on the effects of 3 vitamins, A = nicotinic acid, B = thiamine, and C = biotin, on the yield (\( \frac{1}{100} \text{g/L} \)) of the organic acid pyruvate in a cell culture used two replicates per treatment. In the data, “-1” indicates the low factor level and “1” indicates the high level:

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Treatment</th>
<th>Yields</th>
<th>Mean Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>a</td>
<td>55, 49</td>
<td>51</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>b</td>
<td>37, 28</td>
<td>32.5</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>c</td>
<td>30, 28</td>
<td>32.5</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>ac</td>
<td>54, 54</td>
<td>54</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>bc</td>
<td>54, 47</td>
<td>50.5</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>abc</td>
<td>44, 33</td>
<td>38.5</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td>36, 20</td>
<td>28</td>
</tr>
</tbody>
</table>

Estimate the main effects and interactions. Which do you think are most important?

- \( A = \)
- \( B = \)
- \( C = \frac{1}{4}(50.5 + 28 + 54 + 38.5) - \frac{1}{4}(51 + 29 + 52 + 32.5) = 1.625 \)
- \( AB = \)
- \( AC = \)
- \( BC = \frac{1}{4}(28 + 38.5 + 51 + 52) - \frac{1}{4}(50.5 + 54 + 29 + 32.5) = 0.875 \)
- \( ABC = \)

Next time we’ll see how to test which of these main effects and interactions are ____________.