CS412: Lecture #24

Mridul Aanjaneya

April 28, 2015

Initial value problems for 1st order ordinary differential equations

Problem statement: Find the function $y(t) : [t_0, \infty) \to \mathbb{R}$ that satisfies the ordinary differential equation:

$$y'(t) = f(t, y(t)) \qquad \text{for a certain function } f \tag{1}$$

and $y(t_0) = y_0$.

Method: (1-step algorithms)

- Set $t_k = t_0 + k\Delta t$.
- Define $y_k = y(t_k)$.
- Iteratively approximate

$$\left.\underbrace{y_{k+1} = y_k + \Delta t f(t_k, y_k)}_{\text{(Forward Euler)}}\right\}\text{Explicit}$$

$$\left.\begin{array}{c}\underbrace{y_{k+1} = y_k + \Delta t f(t_{k+1}, y_{k+1})}_{\text{(Backward Euler)}}\\ \underbrace{y_{k+1} = y_k + \frac{\Delta t}{2}[f(t_k, y_k) + f(t_{k+1}, y_{k+1})]}_{\text{(Trapezoidal)}}\end{array}\right\}\text{Implicit}$$

Before we use one of these algorithms in practice we need to examine their limitations and ensure they are usable for a specific problem. We look at the following:

Properties of the ODE itself	Properties of the numerical method
Are the solutions to the ODE *stable*?	Is the numerical method stable? Under what conditions? What is the accuracy of the method?

Stability of solutions to ODE

Under normal circumstances, we expect the ODE (1) to have a unique solution; however, if we omit the initial condition, we get an entire *family* of solutions to the ODE. For example,

- consider $y'(t) = \lambda y(t)$. The exact solution is $y(t) = ce^{\lambda t}$ (for any arbitrary $c \in \mathbb{R}$). For $\lambda > 0$, the solutions diverge because $e^{\lambda t} \to \infty$ as $t \to \infty$, while for $\lambda < 0$ the solutions converge because $e^{\lambda t} \to 0$ as $t \to \infty$.

- consider $y'(t) = f(t)$ (f is a function of t alone *not* y). The exact solution is $y(t) = \int_{t_0}^{t} f(\tau)d\tau + c$, where $c \in \mathbb{R}$ is an arbitrary constant. In this case, the solutions stay a fixed distance apart.

Definition: An ODE is said to have *stable* solutions if the distance between any two solutions y and $\hat{y}$ remains *bounded*, i.e., $|y(t) - \hat{y}(t)| \leq \mathsf{constant}$, $\forall t \geq t_0$. (Strictly speaking, we must also be able to make this constant *arbitrarily small*, by bringing the initial values y_0 and $\hat{y}_0$ close together.)

If additionally, we have that for any two solutions $y(t)$ and $\hat{y}(t)$, we have $\lim_{t\to\infty} |y(t) - \hat{y}(t)| = 0$, the ODE has *asymptotically stable solutions.* Note that if an ODE is asymptotically stable, then it is stable too.

Otherwise (i.e., when the solutions diverge from one another) the ODE is said to have *unstable* solutions.

The ODE $\boxed{y'(t) = \lambda y(t)}$ is called the *model* 1st order ODE and is extremely useful as an example in the analysis of stability, etc. We have:

- When $\lambda < 0$, the solutions to the model ODE are *asymptotically stable* (converge towards one another).

- When $\lambda > 0$, the solutions are *unstable* (diverge away).

- When $\lambda = 0$, the solutions are *stable* although not asymptotically stable (they stay within bounded distance).

For a more general ODE $y' = f(t, y)$, the criteria are:

- If $\frac{\partial f}{\partial y}(t, y) < 0$ for all t and y, the solutions are *asymptotically stable.*

- If $\frac{\partial f}{\partial y}(t, y) \leq 0$ for all t and y, the solutions are *stable* (but not necessarily asymptotically stable).

- If $\frac{\partial f}{\partial y}(t, y)$ is positive or changes sign, we cannot conclude stability with certainty.

Why do we want ODEs with stable solutions?

- Errors (approximation, truncation, roundoff) tend to move us away from the "intended" solution to an IVP, and onto another function from the family of solutions to the ODE. If the solution if stable (or even better, asymptotically stable) then the error remains bounded (or diminishes, for asymptotic stability) over time.
- ODEs with unstable solutions are prone to developing problematic behaviors. For example, different solutions may become *undefined* after a certain (solution-dependent) point in time. For example, $y'(t) = ty^3$, the exact solution is $y(t) = \pm\frac{1}{\sqrt{c-t^2}}$.

Designing a "usable" algorithm for approximating solutions to an unstable ODE is highly nontrivial, and we will not address it in CS412! So, we will continue under the premise that the ODE in question is stable.

Sometimes, even if the ODE is stable, an approximation method may diverge/overflow! For example, $y' = \lambda y$, $\lambda < 0$ (exact solution $y(t) = y_0 e^{\lambda(t-t_0)}$). Using Forward Euler:

$$\begin{aligned} y_{k+1} &= y_k + \Delta t \lambda y_k = (1 + \lambda\Delta t) y_k \\ \Rightarrow y_k &= (1 + \lambda\Delta t)^k y_0 \end{aligned}$$

When $\lambda < 0$, the exact solution satisfies:

$$y(t) = y_0 e^{\lambda(t-t_0)} \xrightarrow{t\to\infty} 0$$

However, for the approximate solution

$$y_k = (1 + \lambda\Delta t)^k y_0 \xrightarrow{t\to\infty} \begin{cases} \text{Converges to } 0 \text{ if } |1+\lambda\Delta t| < 1 \\ \text{Diverges to } \pm\infty \text{ if } |1+\lambda\Delta t| > 1 \\ \text{Oscillates, if } |1+\lambda\Delta t| = 1 \end{cases}$$

Definition: A numerical method is called *stable* when applied to an ODE with stable solutions, if it exhibits the same asymptotic behavior with the exact solution when $t \to \infty$.

In our case, the proper asymptotic behavior is $y_k \xrightarrow{t\to\infty} 0$, which is only guaranteed when

$$\begin{aligned} |1+\lambda\Delta t| < 1 \quad &\Leftrightarrow \quad -1 < 1 + \lambda\Delta t < 1 \\ &\Leftrightarrow \quad -2 < \underbrace{\lambda\Delta t < 0}_{\text{always true since } \lambda < 0} \quad \Leftrightarrow -2 < -|\lambda|\Delta t \\ &\Leftrightarrow \quad \boxed{\Delta t < \frac{2}{|\lambda|}} \leftarrow \text{stability condition for Forward Euler!} \end{aligned}$$

What about Backward Euler?

Again we test on the model stable ODE $y' = \lambda y$, $\lambda < 0$

$$\begin{aligned} y_{k+1} &= y_k + \Delta t f(t_{k+1}, y_{k+1}) \\ &= y_k + \Delta t \cdot \lambda y_{k+1} \\ \Rightarrow (1 - \lambda \Delta t) y_{k+1} &= y_k \\ \Rightarrow y_{k+1} &= \frac{1}{1 - \lambda \Delta t} y_k \Rightarrow \boxed{y_k = \left(\frac{1}{1 - \lambda \Delta t} \right)^k y_0} \end{aligned}$$

Here, on order to have $y_k \xrightarrow{k \to \infty} 0$, we need

$$\left| \frac{1}{1 - \lambda \Delta t} \right| < 1 \Leftrightarrow |1 - \lambda \Delta t| > 1 \leftarrow \textsf{Always true since } \lambda < 0!$$

Thus, Backward Euler is *unconditionally* stable!

Trapezoidal rule

Using the rule on the model ODE $y' = \lambda y$, $\lambda < 0$

$$\begin{aligned} y_{k+1} &= y_k + \frac{\Delta t}{2} [f(t_k, y_k) + f(t_{k+1}, y_{k+1})] \\ &= y_k + \frac{\Delta t}{2} [\lambda y_k + \lambda y_{k+1}] \\ \Rightarrow \left(1 - \frac{\lambda \Delta t}{2}\right) y_{k+1} &= \left(1 + \frac{\lambda \Delta t}{2}\right) y_k \\ \Rightarrow y_k &= \left(\frac{1 + \lambda \Delta t/2}{1 - \lambda \Delta t/2} \right)^k y_0 \end{aligned}$$

For stability, we need:

$$\begin{aligned} \left| \frac{1 + \lambda \Delta t/2}{1 - \lambda \Delta t/2} \right| < 1 \quad &\Leftrightarrow \quad \left| 1 + \frac{\lambda \Delta t}{2} \right| < \underbrace{\left| 1 - \frac{\lambda \Delta t}{2} \right|}_{>0} \Leftrightarrow \left| 1 + \frac{\lambda \Delta t}{2} \right| < 1 - \frac{\lambda \Delta t}{2} \\ &\Leftrightarrow \quad -1 + \frac{\lambda \Delta t}{2} < 1 + \frac{\lambda \Delta t}{2} < 1 - \frac{\lambda \Delta t}{2} \end{aligned}$$

The left inequality is always true, while the right inequality is always true for $\lambda < 0$. Thus, the trapezoidal rule is also unconditionally stable!