CS515 Spring 2008

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Assignment # 2 answer key

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1. To show that λ is a linear functional we need to check additivity and homogenuity.
(a) λ is a linear functional:

$$\begin{aligned}\lambda(f+g) &= (f+g)(t_0) = f(t_0) + g(t_0) = \lambda(f) + \lambda(g) \\ \lambda(cf) &= (cf)(t_0) = c\, f(t_0) = c\, \lambda(f)\end{aligned}$$

(b) λ is a linear functional:

$$\begin{aligned}\lambda(f+g) &= (f+g)''(t_0) = f''(t_0) + g''(t_0) = \lambda(f) + \lambda(g) \\ \lambda(cf) &= (cf)''(t_0) = c\, f''(t_0) = c\, \lambda(f)\end{aligned}$$

(c) λ is a linear functional:

$$\begin{aligned}\lambda(f+g) &= (f+g)(t_0) - (f+g)'(t_1) = f(t_0) - f'(t_1) + g(t_0) - g'(t_1) = \lambda(f) + \lambda(g) \\ \lambda(cf) &= (cf)(t_0) - (cf)'(t_1) = c\, f(t_0) - c\, f'(t_1) = c(f(t_0) - f'(t_1)) = c\, \lambda(f)\end{aligned}$$

(d) λ is not a linear functional.
Counterexample:

$$\lambda(2f) = (2f)(t_0)\,(2f)(t_1) = 4f(t_0)f(t_1) \neq 2\lambda(f).$$

(e) λ is not a linear functional (except when $t_0 = t_1$).
Counterexample: Choose $f(t) \equiv 1,\ g(t) \equiv -1$. Then

$$\begin{aligned}\lambda(f+g) &= \int_{t_0}^{t_1} |1-1|dt = 0 \\ \lambda(f) + \lambda(g) &= \int_{t_0}^{t_1} |1|dt + \int_{t_0}^{t_1} |-1|dt = 2|t_1 - t_0|.\end{aligned}$$

(f) λ is a linear functional:

$$\begin{aligned}\lambda(f+g) &= \int_{t_0}^{t_1} (f+g)(t)\, t^2 dt = \int_{t_0}^{t_1} f(t)\, t^2 dt + \int_{t_0}^{t_1} g(t)\, t^2 dt = \lambda(f) + \lambda(g) \\ \lambda(cf) &= \int_{t_0}^{t_1} (cf)(t)\, t^2 dt = c \int_{t_0}^{t_1} f(t)\, t^2 dt = c\, \lambda(f).\end{aligned}$$

(g) λ is not a linear functional.
Counterexample: Choose

$$f(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2}, \\ 0, & \frac{1}{2} < t \le 1, \end{cases} \qquad g(t) = \begin{cases} -1, & 0 \le t \le \frac{1}{2}, \\ 0, & \frac{1}{2} < t \le 1, \end{cases}$$

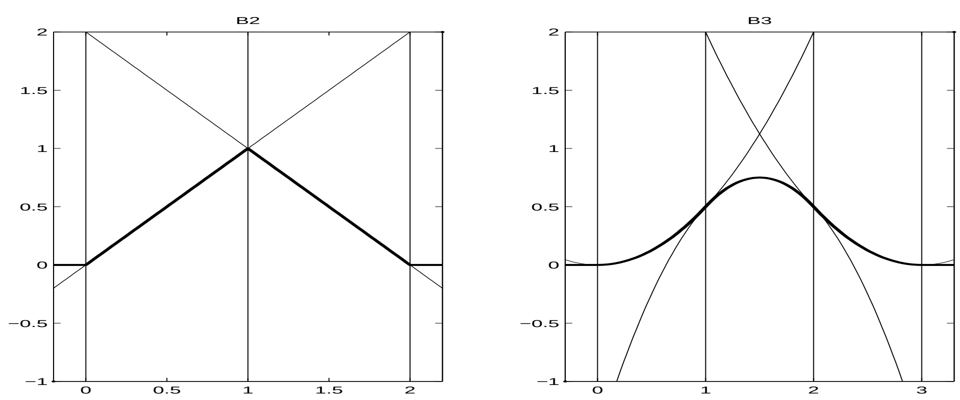
Then

$$\lambda(f+g) = 0 \neq 1 = 1 + 0 = \lambda(f) + \lambda(g)$$

2.

$$\begin{aligned}
B_2(t) &= \int_{\mathbb{R}} B_1(u)\, B_1(t-u)du = \int_0^1 B_1(t-u)du \\
&= \begin{cases} 0, & t \leq 0 \ (t-u \notin [0,1]), \\ \int_0^t du, & 0 < t \leq 1 \text{ (to have } 0 < t-u \text{ we need } u < t), \\ \int_{t-1}^1 du, & 1 < t \leq 2 \text{ (to have } t-u < 1 \text{ we need } u > t-1), \\ 0, & 2 < t \ (t-u \notin [0,1]), \end{cases} \\
&= \begin{cases} 0, & t \leq 0, \\ t, & 0 < t \leq 1, \\ 2-t, & 1 < t \leq 2, \\ 0, & 2 < t. \end{cases}
\end{aligned}$$

$$\begin{aligned}
B_3(t) &= \int_{\mathbb{R}} B_2(u) B_1(t-u)du = \int_0^1 u\, B_1(t-u)du + \int_1^2 (2-u)\, B_1(t-u)du \\
&= \begin{cases} 0, & t \leq 0, \\ \int_0^t udu, & 0 < t \leq 1, \\ \int_{t-1}^1 udu + \int_1^t (2-u)du, & 1 < t \leq 2, \\ \int_{t-1}^2 (2-u)ds, & 2 < t \leq 3, \\ 0, & 3 \leq t, \end{cases} = \begin{cases} 0, & t \leq 0, \\ \frac{t^2}{2}, & 0 < t \leq 1, \\ -t^2 + 3t - \frac{3}{2}, & 1 < t \leq 2, \\ \frac{(3-t)^2}{2}, & 2 < t \leq 3, \\ 0, & 3 < t. \end{cases}
\end{aligned}$$



3.

$$f * g(t) := \int_{-\infty}^{\infty} f(u) g(t-u) ds.$$

Make a substitution $s = t - u$. Then

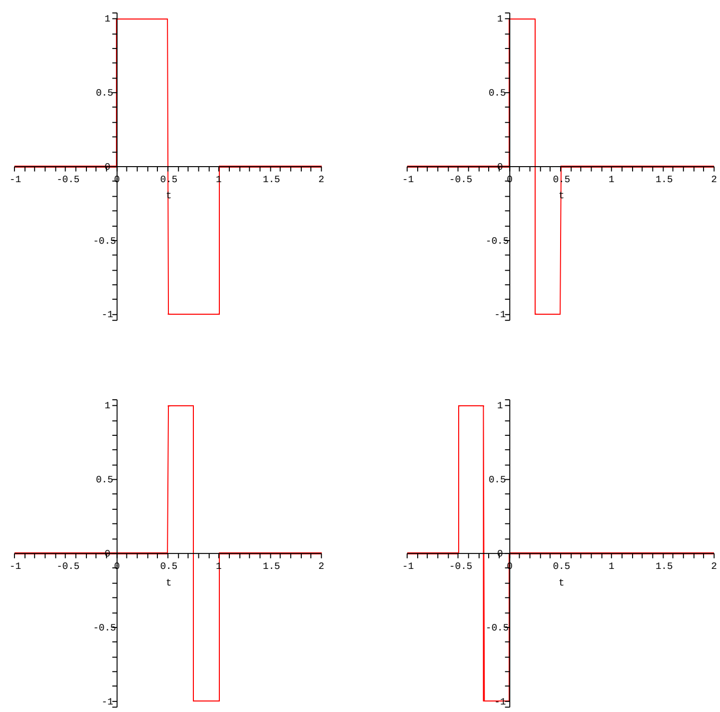
$$f * g(t) = -\int_{\infty}^{-\infty} f(t-s)g(s)ds = \int_{-\infty}^{\infty} g(s)f(t-s)ds = g * f(t).$$

4. (a) Fourier coefficients $\hat{f}(k)$ of f are

$$\hat{f}(k) = \int_{-\pi}^{\pi} te^{-ikt}dt = \frac{2\pi i \cos k\pi}{k}.$$

The Fourier coefficient at $k = 0$ should be calculated separately, and can be shown to be 0.
(b) The function f is not continuous as a periodic function, therefore its Fourier coefficients decay slowly.

5. (a)$H, H_{1,0}, H_{1,1}$, and $H_{1,-1}$



(b) $H_{1,-1}, H_{1,0}$, and $H_{1,1}$ are mutually orthogonal because they have disjoint supports. For the same reason, $H_{1,-1}$ is orthogonal to H. Now,

$$\langle H, H_{1,0} \rangle = \int_{\mathbb{R}} H(t)\overline{H_{1,0}(t)}dt = \int_{0}^{1/4} dt + \int_{1/4}^{1/2} (-1)dt = 0,$$

and

$$\langle H, H_{1,1}\rangle = \int_{\mathbb{R}} H(t)\overline{H_{1,1}(t)}dt = \int_{1/2}^{3/4} dt + \int_{3/4}^{1} (-1)dt = 0.$$

(c)

$$\langle H, H\rangle = \int_0^1 dt = 1,$$

so H does not require normalization.

$$\langle H_{1,0}, H_{1,0}\rangle = \int_0^{1/2} dt = 1/2,$$

so $H_{1,0}$ has to be multiplied by $\sqrt{2}$ in order to be normalized. Since $H_{1,-1}$ and $H_{1,1}$ are shifts of $H_{1,0}$, they have to be normalized by the same factor.