CS515 Spring 08
Prof. Ron

Assignment #4: Part I

Due February 21, 2008

The assignment is made of two parts. The first part deals with wavelets and vanishing moments. The rest of the assignment is a `matlab` tutorial followed by a few additional questions and is separate file.

Question # 1.
The function $f \in L_2(\mathbb{R})$ is supported in the interval $[0, 1]$ and has m continuous derivatives, with m some non-negative integer. ψ is a (single) mother wavelet, is supported in $[0, 4]$ and has m vanishing moments. The wavelet system $X(\psi)$ is known to satisfy the perfect reconstruction property. Therefore,

$$f = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi_{j,k}.$$

Instead of expanding fully f as above, we decide to truncate the above sum at some dilation level j_0:

$$f_{j_0} := \sum_{j=-\infty}^{j_0} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi_{j,k}.$$

Our hope is that f_{j_0} is a good approximation to f. Thus, you need to provide estimates on

$$\|f - f_{j_0}\|.$$

A good estimate should take the form $\|f - f_{j_0}\| \leq C2^{-j_0\alpha}$, with α as large as possible, and with C independent of j_0. Derive two such estimates, using the following guidelines.

First, note that

$$f - f_{j_0} = \sum_{j=j_0+1}^{\infty} \sum_{k=-\infty}^{\infty} \langle f, \psi_{j,k} \rangle \psi_{j,k}.$$

Fix a dilation level j. Then most of the coefficients at level j are trivially zero. Find the exact number of terms that are not guaranteed to be zero, and estimate each one of them using the information about f and ψ.

Then assume first that $X(\psi)$ is a complete orthonormal basis for $L_2(\mathbb{R})$ (this will be give you the perfect reconstruction for free). This condition implies that

$$\|f - f_{j_0}\|^2 = \sum_{j=j_0+1}^{\infty} \sum_{k=-\infty}^{\infty} |\langle f, \psi_{j,k} \rangle|^2.$$

Substitute your bounds on the inner products and sum up. This way you get your best estimate on the error.

If $X(\psi)$ is not orthonormal, you can still use the so-called triangle inequality which tells you that

$$\|f - f_{j_0}\| \leq \sum_{j=j_0+1}^{\infty} \sum_{k=-\infty}^{\infty} |\langle f, \psi_{j,k} \rangle| \|\psi_{j,k}\|.$$

This will give you a second estimate that is slightly worse than the first one.

Question # 2.

The cubic B-spline $B_4 = B_1 * B_1 * B_1 * B_1$ is obtained by convolving B_1 with itself 3 times (Note: you are not asked to compute B_4). This implies (how?) that

$$\widehat{B_4}(\omega) = \left(\frac{1 - \mathrm{e}^{-i\omega}}{i\omega}\right)^4.$$

Show that B_4 is refinable and find its refinement mask (write the refinement mask as a trigonometric polynomial, as well as a sequence defined on the integers).