cs 514, lecture 10apr00 – ??: B-splines defined

The traditional definition of a B-spline is in terms of the divided difference of the truncated power function

$$()^j_+ : x \mapsto x^j_+ := \begin{cases} x^j, & x \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

However, since I did not take the time to introduce divided differences (so far), I'll introduce B-splines instead 'out of the blue' via their recurrence, with the B-spline of order 1 the starting point for that recurrence.

Definition. *The* **B-spline with knots** a, b *is the characteristic function of the half-open interval* $[a\,..\,b)$*, i.e.,*

$$B(x|a,b) := \chi_{[a..b)}(x) = \begin{cases} 1, & a \le x < b; \\ 0, & \textit{otherwise} \end{cases}.$$

Observations

- $B(\cdot|a,b)$ is piecewise constant, with a breaks a and b (and nowhere else), in case $a < b$; in particular, it is positive on the interval $[a\,..\,b)$, and is zero elsewhere.
- $B(\cdot|a,b)$ is the zero function in case $a \ge b$.
- $B(\cdot|a,b)$ is **right-continuous**, i.e.,

$$B(x|a,b) = B(x^+|a,b) := \lim_{h \downarrow 0} B(x+h|a,b).$$

The choice of making the B-spline right-continuous is just that, a choice, since some choice has to be made, and other choices would have been possible, e.g., the more symmetric choice

$$f(x) := (f(x-) + f(x+))/2.$$

The particular choice made is connected to the ppform, a standard way to represent piecewise polynomials.

If now

$$\mathbf{t} := (\cdots \le t_i \le t_{i+1} \le \cdots)$$

is a given nondecreasing sequence, we associate with it the sequence

$$B_i = B_{i,1,\mathbf{t}} := B(\cdot|t_i, t_{i+1}), \quad \forall i,$$

of first-order B-splines, and find that this sequence is a **partition of unity**, i.e.,

$$\sum_i B_i(x) = 1, \quad \inf_i t_i < x < \sup_i t_i.$$

Further, if $t_i < t_{i+1}$ for all i, then $(B_i : i)$ provides a 'basis' for the space

$$\Pi_{<1,\mathbf{t}}$$

of all (right-continuous) piecewise constant functions with possible breaks at the t_i's and nowhere else. Here, I have put 'basis' in quotes since I don't exclude the possibility that the sequence $\mathbf{t}$ is infinite or even bi-infinite, hence correspondingly consider the infinite or even bi-infinite sum

$$f = \sum_i a(i) B_i : x \mapsto \sum_i a(i) B_i(x) = \begin{cases} a(j), & t_j \le x < t_{j+1}; \\ 0, & \text{otherwise,} \end{cases}$$

i.e., as a pointwise sum.

In practice, one usually only considers finite sequences $\mathbf{t}$, but it is convenient to permit $\mathbf{t}$ to be nonfinite. In fact, in your consideration of B-splines so far, the sequence $\mathbf{t}$ was always bi-infinite, namely the sequence $\mathbf{t} = \mathbb{Z}$.

Definition. *With* $\mathbf{t}$ *continuing to denote a nondecreasing sequence,*

$$B_i := B_{i,2} := B_{i,2,\mathbf{t}} := B(\cdot|t_i, t_{i+1}, t_{i+2})x \mapsto \begin{cases} \frac{x-t_i}{t_{i+1}-t_i}, & t_i \le x < t_{i+1} \\ \frac{t_{i+2}-x}{t_{i+2}-t_{i+1}}, & t_{i+1} \le x < t_{i+2} \\ 0, & \text{otherwise,} \end{cases}$$

is the **B-spline with knots** t_i, t_{i+1}, t_{i+2}.

Observations

- $B_{i,2} = B(\cdot|t_i, t_{i+1}, t_{i+2})$ is piecewise linear, with breaks at t_i, t_{i+1}, t_{i+2}; it is positive on $(t_i \mathinner{.\,.} t_{i+2})$ and is zero off $[t_i \mathinner{.\,.} t_{i+2}]$.
- $B_{i,2}$ is continuous from the right and it continuous at any of its breaks if and only if that break appears only once the the sequence $\mathtt{t}$.

It is still true that

$$\sum_i B_{i,2}(x) = 1, \quad \inf_i t_i < x < \sup_i t_i,$$

except for some possible trouble near the first or last t_i, if there is one. It is a little bit harder, though, to describe precisely the span of $(B_{i,2} : i)$.

Now the following observation: *With* $\omega_{i,2}$ *the linear polynomial*

$$\omega_{i,2} : x \mapsto \frac{x - t_i}{t_{i+1} - t_i}, \quad \forall i,$$

we have

$$B_{i,2} = \omega_{i,2} B_{i,1} + (1 - \omega_{i+1,2}) B_{i+1,1}.$$

Thus $B_{2,i}$ fits the following **recursive definition** of the B-spline.

Definition. *With* $\mathbf{t}$ *a nondecreasing sequence and* k *a positive integer, the B-spline with knots* $t_i, \ldots, t_{i+k}$ *is*

$$B(\cdot|t_i, \ldots, t_{i+k}) := B_{i,k} := B_{i,k,\mathbf{t}} := \begin{cases} \chi_{[t_i \ldots t_{i+1})}, & k = 1; \\ \omega_{i,k} B_{i,k-1} + (1 - \omega_{i+1,k}) B_{i+1,k-1}, & k > 1, \end{cases}$$

where $\omega_{i,k}$ *is the linear polynomial*

$$\omega_{i,k} : x \mapsto \frac{x - t_i}{t_{i+k-1} - t_i}.$$

Use the `MATLAB Spline Toolbox` command `bspligui` to become more familiar with just how $B_{i,k,\mathtt{t}}$ depends on its $k+1$ knots.

Also, marvel at the fact that the recurrence relation takes two functions of a certain smoothness, multiplies each of them by some linear polynomial, and then adds the resulting products and, in this way, obtains a function with higher smoothness than either summand.

The discussion of B-spline properties to follow is taken from the book (though I won't necessarily go through the proofs), hence I won't prepare class notes for it.

