cs 514, lecture 14apr00 – ??: B-spline recurrence used

Let

$$f := \sum_j a_j B_{jk}.$$

If $k > 1$, then, from the recurrence relations,

$$\begin{aligned} f &= \sum_j a_j(\omega_{jk} B_{j,k-1} + (1-\omega_{j+1,k})B_{j+1,k-1}) \\ &= \sum_j \left(a_j\omega_{jk} + a_{j-1}(1-\omega_{j,k})\right) B_{j,k-1}. \end{aligned}$$

Hence, with

$$a_j^{[i+1]} := \begin{cases} a_j, & i = 0; \\ a_j^{[i]}\omega_{j,k-i+1} + a_{j-1}^{[i]}(1-\omega_{j,k-i+1}) = \dfrac{(\cdot - t_j)a_j^{[i]} + (t_{j+k-i} - \cdot)a_{j-1}^{[i]}}{t_{j+k-i} - t_j}, & i > 0, \end{cases}$$

we get

$$f = \sum_j a_j^{[i]} B_{j,k-i+1}, \qquad i = 1{:}k.$$

In particular,

$$f = \sum_j a_j^{[k]} B_{j,1},$$

with each $a_j^{[k]}$ a polynomial of degree $< k$, i.e., a polynomial of **order** k,

$$a_j^{[k]} \in \Pi_{<k}.$$

Consider some specific sequences $a = (a_j : j)$.

1. $a = \delta_j$, i.e., $f = B_{jk}$. Then, by induction, $a_j^{[i]}, \ldots, a_{j+i-1}^{[i]}$ are the only nonzero entries in $a^{[i]}$. In particular, B_{jk} has its support in the interval $[t_j \,..\, t_{j+k})$.

2. $a_j = 1$, all j. Then also $a_j^{[i]} = 1$ for all j, therefore

$$\sum_j B_{jk} = \sum_j B_{j1} = 1,$$

showing that $(B_{jk} : j)$ forms a (positive and local) partition of unity (B-spline property (ii)).

3. This example is the prettiest:

$$a_j = (t_{j+1} - \tau)\cdots(t_{j+k-1} - \tau) =: \psi_{jk}(\tau), \quad \forall j,$$

with τ arbitrary. Then

$$\begin{aligned} a_j^{[2]} &= \psi_{jk}(\tau)\omega_{jk} + \psi_{j-1,k}(\tau)(1-\omega_{jk}) \\ &= \psi_{j,k-1}(\tau)\left((t_{j+k-1} - \tau)\omega_{jk} + (t_j - \tau)(1-\omega_{jk})\right) \\ &= \psi_{j,k-1}(\tau)(\cdot - \tau). \end{aligned}$$

In other words,

$$\sum_j \psi_{jk}(\tau)B_{jk} = (\cdot - \tau)\sum_j \psi_{j,k-1}(\tau)B_{j,k-1} = \cdots = (\cdot - \tau)^{k-1}\sum_j \psi_{j,1}(\tau)B_{j,1} = (\cdot - \tau)^{k-1}.$$

This is **Marsden's identity** (B-spline property (iii)):

$$(\cdot - \tau)^{k-1} = \sum_j \psi_{jk}(\tau) B_{jk}.$$

From this, we get a formula for writing any $p \in \Pi_{<k}$ as a weighted sum of the B_{jk} by going through the following steps:

1. divide both sides by $(k-1)!$ (to make differentiation easier)
2. differentiate both sides $\nu - 1$ times with respect to $-\tau$
3. multiply both sides by $D^{k-\nu}p(\tau)$
4. sum all these equations for $\nu = 1, \ldots, k$.

This gives the (horrendous) equation

$$\sum_{\nu=1}^{k} D^{k-\nu}p(\tau)\frac{(\cdot - \tau)^{k-\nu}}{(k-\nu)!} = \sum_{\nu=1}^{k} D^{k-\nu}p(\tau) \sum_j \frac{(-D)^{\nu-1}\psi_{jk}(\tau)}{(k-1)!} B_{jk}.$$

However, since we assume that p is of degree $< k$, the left side is just the Taylor expansion for p (around the point τ), hence equal to p. Further, since we agreed that the sum over j is to be taken pointwise and, at any point x, at most k of the $B_{jk}(x)$ are not zero, there is no difficulty with interchanging those two summations. After that, we finally get that

$$p = \sum_j \lambda_{jk} p\, B_{jk}, \qquad \forall p \in \Pi_{<k}, \tag{1}$$

with

$$\lambda_{jk} : f \mapsto \sum_{\nu=1}^{k} \frac{(-D)^{\nu-1}\psi_{jk}(\tau)}{(k-1)!} D^{k-\nu} f(\tau). \tag{2}$$

Take in the fact that this holds for an arbitrary τ. (In fact, it is easy to verify that, *for any* $f \in \Pi_{<k}$, $\lambda_{jk}f$ *is independent of* τ).

As a quick check, take for p a constant polynomial. Then all derivatives of p are zero everywhere, hence

$$\lambda_{jk}p = \frac{(-D)^{k-1}\psi_{jk}(\tau)}{(k-1)!} p(\tau),$$

and this equals $p(\tau)$ since $\psi_{jk}(\tau) = (-\tau)^{k-1} + \text{l.o.t.}$, hence $(-D)^{k-1}\psi_{jk} = (k-1)!$. We conclude that

$$\lambda_{jk}p = p(\tau), \qquad p \in \Pi_0.$$

A more interesting case occurs when p is a linear polynomial, say $p = \ell \in \Pi_1$. Now $D^i p(\tau) = 0$ for any $i > 1$. Therefore,

$$\lambda_{jk}\ell = \ell(\tau) + \frac{(-D)^{k-2}\psi_{jk}(\tau)}{(k-1)!} D\ell(\tau).$$

Now, since ψ_{jk} is a polynomial of exact degree $k-1$, its $(k-2)$nd derivative is a polynomial of exact degree 1, hence has exactly one zero. This zero turns out to be the point

$$t^*_{jk} := (t_{j+1} + \cdots + t_{j+k-1})/(k-1).$$

In other words

$$\lambda_{jk}\ell = \ell(t^*_{jk}), \qquad \forall \ell \in \Pi_1,$$

hence (B-spline property (v))

$$\ell = \sum_j \ell(t^*_{jk}) B_{jk}, \qquad \forall \ell \in \Pi_1.$$

It turns out that the formula (1) holds not just for $p \in \Pi_{<k}$, but for every $p = \sum_j a_j B_{jk}$ with arbitrary coefficient sequence $(a_j : j)$, provided only that the τ appearing in the definition (2) of the linear functional λ_{jk} be, more precisely, some point τ_j in the support of B_{jk}, i.e., from the interval $(t_j \mathinner{\ldotp\ldotp} t_{j+k})$.

