cs 514, lecture 17apr00 – ??: spline := linear combination of B-splines

By definition, *a spline of order k with knot sequence* $\mathbf{t}$ is a weighted sum of the B-splines $B_{j,k,\mathbf{t}}$, i.e., an element of

$$\$:= \$_{k,\mathbf{t}} := \operatorname{ran}(B_{j,k,\mathbf{t}} : j) = \{\sum_j a_j B_{j,k,\mathbf{t}} : a_j \in \mathbb{R}\}.$$

Exactly what functions are in $\$_{k,\mathbf{t}}$?

Pick a nontrivial knot interval,

$$I_j := (t_j \mathinner{..} t_{j+1}) \neq \emptyset,$$

say. Then, on such an interval, every $B_i = B_{i,k,\mathbf{t}}$ is just a polynomial of order k, i.e., of degree $< k$, hence every $f \in \$_{k,\mathbf{t}}$ has the same property:

$$\$_{k,\mathbf{t}}\big|_{I_j} \subset \Pi_{<k}\big|_{I_j}.$$

Is there equality? That depends. For each i, let p_i be the polynomial that agrees with B_i on I_j. Then at most k of these are nonzero, namely $p_{j+1-k}, \ldots, p_j$, hence, since $\dim \Pi_{<k} = k$, there cannot be equality unless all k of these p_i actually exist. Note that there could be fewer. If, e.g., $\mathbf{t}$ has a first entry, $\mathbf{t} = (t_1, t_2, \ldots)$, say, then, for $j = 1$, only p_j has a chance of being defined. For that reason, I work with the **basic interval** for $\$_{k,\mathbf{t}}$ (see B-spline properties hand-out)

$$I_{k,\mathbf{t}} := (\mathbf{t}_- \mathinner{..} \mathbf{t}_+),$$

with

$$\mathbf{t}_- := \begin{cases} t_k, & \text{if } \mathbf{t} = (t_1, \ldots); \\ \inf_j t_j, & \text{otherwise,} \end{cases} \qquad \mathbf{t}_+ := \begin{cases} t_{n+1}, & \text{if } \mathbf{t} = (\ldots, t_{n+k}); \\ \sup_j t_j, & \text{otherwise.} \end{cases}$$

For each $I_j \subset I_{k,\mathbf{t}}$, the full sequence $(p_{j+1-k}, \ldots, p_j)$ is defined. Moreover, we already know that

$$p = \sum_{i=j+1-k}^{j} (\lambda_{ik} p)\, p_i, \qquad \forall p \in \Pi_{<k}.$$

Since there are exactly k of these p_i involved here, and $\dim \Pi_{<k} = k$, we conclude that $[p_{j+1-k}, \ldots, p_j]$ is a basis for $\Pi_{<k}$ and that

$$\lambda_{ik} p_h = \delta_{ih}, \qquad i, h = j+1-k, \ldots, j,$$

i.e., the data map $\Pi_{<k} \to \mathbb{R}^k : p \mapsto (\lambda_{ik} p : i = j+1-k, \ldots, j)$ is its inverse.

The linear functionals

$$\lambda_{ik} : f \mapsto \sum_{\nu=1}^{k} \frac{(-D)^{\nu-1} \psi_{ik}(\tau_i)}{(k-1)!} D^{k-\nu} f(\tau_i) \tag{1}$$

involved here depend on the choice of the points τ_i. As already remarked, $\lambda_{ik} f$ is independent of the choice of τ_i in case $f \in \Pi_{<k}$. But this independence cannot hold for every $f \in \$_{k,\mathbf{t}}$: E.g., if $f = B_{ik}$ for $i = j+1-k, \ldots, j$, then we just saw that $\lambda_{ik} f = 1$ if $\tau_i \in I_j$, while it is certainly 0 in case τ_i is outside the interval $(t_i \mathinner{..} t_{i+k})$. Remarkably, that is the *only* restriction. In particular, assuming from now on that

$$t_i < \tau_i < t_{i+k}, \quad \forall i,$$

we have

$$\lambda_{ik} B_{jk} = \delta_{ij}, \quad \forall i, j.$$

For this reason, the λ_{ik} are called the **dual functionals** (for the corresponding B-spline sequence). See B-spline property (vii).

In particular $(B_{jk} : j)$ is linearly independent, hence a basis for its span, $\$_{k,\mathbf{t}}$. It is for this reason that their creator, I. J. Schoenberg, gave them the letter 'B', as an acronym for 'Basis' or 'basic'.

By now, we know that each $f \in \$_{k,\mathbf{t}}$ is piecewise polynomial of order k with breaks at the t_i (and nowhere else). For a fuller description, return once more to Marsden's Identity:

$$(\cdot - \tau)^{k-1} = \sum_j \psi_{jk}(\tau) B_{jk},$$

but choose $\tau = t_i$ for some i, and observe that

$$\psi_{jk}(t_i) B_{jk}(t_i) = 0 \quad \forall i.$$

Indeed, if $B_{jk}(t_i) \neq 0$, then necessarily $t_j < t_i < t_{j+k}$, hence t_i is one of the zeros of ψ_{jk}. Hence,

$$(\cdot - t_i)^{k-1} = \sum_{B_{jk}(t_i)=0} \psi_{jk}(\tau) B_{jk}.$$

This sum neatly splits into two, one involving all the B-splines with support to the left of t_i, the other involving all the B-splines with support to the right of t_i. In particular,

$$(\cdot - t_i)_+^{k-1} = \sum_{j \geq i} \psi_{jk}(\tau) B_{jk}.$$

So, in addition to $\Pi_{<k}$, $\$_{k,\mathbf{t}}$ also contains all the truncated powers

$$(\cdot - t_i)_+^{k-1}, \qquad t_i < \mathbf{t}_+.$$

One shows in a similar way that, more generally, $\$_{k,\mathbf{t}}$ contains all the truncated powers

$$(\cdot - t_i)_+^{k-\nu}, \qquad 1 \leq \nu \leq \#t_i := \#\{j : t_i = t_j\},\ t_i < \mathbf{t}_+.$$

Note here the appearance of the **knot multiplicity** $\#t_i$ which counts the number of times the number t_i appears in the knot sequence $\mathbf{t}$.

And that's it, as is made clear in B-spline property (vi). Specifically, if you take any interval $I := [a \,..\, b] \subset I_{k,\mathbf{t}}$, then we now know that the entire sequence

$$(((\cdot - a)^{k-\nu} : \nu = 1{:}k), ((\cdot - t_i)_+^{k-\nu} : 1 \leq \nu \leq \#t_i, a < t_i < b))|_I$$

of $k + \#\{i : a < t_i < b\}$ functions is in $\$_{k,\mathbf{t}}|_I$ and is also easily seen to be linearly independent. On the other hand, $\$_{k,\mathbf{t}}|_I$ is spanned by the sequence $(B_{jk}|_I \neq 0)$ and this sequence also contains exactly $k + \#\{i : a < t_i < b\}$ functions. Hence, Linear Algebra tells us that both sequences must be a basis for $\$_{k,\mathbf{t}}|_I$. In particular, *each $f \in \$_{k,\mathbf{t}}$ satisfies (at least) $k - \#t_i$ smoothness conditions across the knot t_i, all i*, i.e., $\mathrm{jump}_{t_i} D^{r-1} f = 0$ for $r = 1{:}(k - \#t_i)$:

knot multiplicity + smoothness = order

Next: What information about $f = \sum_j a_j B_{jk}$ is 'easily' obtained from its B-spline coefficients $(a_j : j)$?

1. evaluation: . To compute $f(x)$, (i) determine j such that $t_j \leq x < t_{j+1}$ and initialize $b := (a_{j+1-k}, \ldots, a_j)$; then (ii) use the recurrence:

```
 for i=2:k
   for r=k:-1:i
      b(r) = ((x-t(j-k+r))*b(r) + (t(j+r-i+1)-x)*b(r-1))/...
              ((x-t(j-k+r))       + (t(j+r-i+1)-x)        );
   end
end
```


After this, $b(k)$ contains the value of f at x. Note that the index for the inner loop runs down rather than up (why?). To be sure, a preferable implementation would compute the quantities `x-t(i)` and `t(k+i)-x`, `i=1:k`, needed here outside the double loop, in which case computation of the denominator is no more costly than in its simpler form `-t(j-k+r) + t(j+r-i+1)`. The present form is preferable for rounding-error control.

What about $Df(x), D^2f(x), \ldots$? Simply 'differentiate' the above double loop, generating with each quantity also its derivatives with respect to x. For example, if we also want the first derivative, we make b a matrix of size `[2,k]`, with $b(2,i)$ the derivative with respect to x of the quantity in $b(1,r)$, $r = 1{:}k$. This means that we initialize `b` as the 2-row matrix

$$b = \begin{bmatrix} a_{j-k+1} & \cdots & a_j \\ 0 & \cdots & 0 \end{bmatrix}$$

(since the coefficients are just constants as far as x is concerned).

The only line of code explicitly involving x is easily differentiated wrto x (note that the denominator is, in fact, independent of x) and so gives the enlarged code:

```
 for i=2:k
   for r=k:-1:i
      b(2,r) = (b(1,r)-b(1,r-1) + ...
                 (x-t(j-k+r))*b(2,r) + (t(j+r-i+1)-x)*b(2,r-1))/...
                 ((x-t(j-k+r))        + (t(j+r-i+1)-x)        );
      b(1,r) = ((x-t(j-k+r))*b(1,r) + (t(j+r-i+1)-x)*b(1,r-1))/...
                 ((x-t(j-k+r))         + (t(j+r-i+1)-x)        );
   end
end
```

Note that the differentiated line is executed first, thus updating the derivative values before updating the function values.

What if x does not lie in the interval $[t_j \,..\, t_{j+1})$? Then $b(k)$ (or, more generally, $b(1,k)$) contains the value at x of the polynomial that agrees with f on that interval.

2. Differentiation (B-spline property (viii)) The derivative of a spline $f = \sum_j a_j B_{jk}$ is a spline of one order lower, and its coefficients are difference quotients of the coefficients of the spline itself:

$$D\big(\sum_j a_j B_{jk}\big) = \sum_j \frac{a_j - a_{j-1}}{(t_{j+k-1} - t_j)/(k-1)} B_{j,k-1}.$$

To be sure, if, e.g., $t_{j+k-1} = t_j$, then that quotient multiplying $B_{j,k-1}$ is not defined. However, in that case, $B_{j,k-1}$ is the zero function, and we don't care.

Note that, in this case, $\#t_j \ge k$, i.e., f itself may have a jump discontinuity across t_j, and is not even differentiable at t_j. In effect, we ignore that, by taking the derivative here piecewise-polynomial style, i.e., for each polynomial piece separately.

As a consequence, $\int_x^y (Df)(s)\,\mathrm{d}s$ will equal $f(y) - f(x)$ in general only if the spline f is continuous on the interval $[x \,..\, y]$, for example if $\#t_i < k$ for all $t_i \in (x \,..\, y)$.

3. Good condition aka stable basis (B-spline property (x)) We already saw that, for $t_j \le x < t_{j+1}$, the value $f(x) = \sum_{i=j-k+1}^{j} a_i B_{ik}(x)$ is a *convex* combination of the k coefficients a_{j-k+i}, $i = 1{:}k$. In particular, the value $f(x)$ must lie between the smallest and the largest of these k coefficients. On the other hand, at least for modest k, none of these k coefficients can be too far from the value $f(x)$. Precisely,

$$|a_i| \le D_{k,\infty} \|\sum_j a_j B_{jk}\|_{[t_{i+1}..t_{i+k-1}]},$$

with $D_{k,\infty} \approx 2^{k-3/2}$.

This makes (B_{jk}) a **stable** basis (or Riesz basis) in the uniform norm in the sense that

$$(1/D_{k,\infty})\|a\|_\infty \le \|\sum_j a_j B_{jk}\|_\infty \le \|a\|_\infty.$$

But the B-spline basis has this property even *locally*.

