cs 514, lecture 21apr00 – ??: B-splines are refinable

The close connection between the value $f(x)$ of $f = \sum_j a_j B_{jk}$ and the 'nearby' coefficients $(a_i : B_{ik}(x) \neq 0)$ is made visible in CAGD by considering the *curve*

$$x \mapsto (x, f(x)) = (\sum_j t^*_{jk} B_{jk}, \sum_j a_j B_{jk}) =: \sum_j P_j B_{jk}$$

(note the use of B-spline property (v) here!), with

$$P_j = P_{j,k,\mathbf{t}} f := (t^*_{jk}, a_j) \in \mathrm{I\!R}^2$$

called the **control point**s, and the broken line connecting these control points, and denoted here by

$$C_{k,\mathbf{t}} f,$$

called the **control polygon**.

This nomenclature arose in CAGD (:= Computer-Aided Geometric Design), where one considers, more generally, **spline curves**, i.e., curves of the form $x \mapsto \sum_{jk} P_j B_{jk}(x)$ with P_j arbitrary vectors in the plane (or even in 3-space or higher dimensions) and, correspondingly, its control polygon, i.e., the piecewise linear curve $x \mapsto \sum_j P_j B_{jk}(x)$.

The control polygon provides a rough outline or caricature of the spline itself. At the same time, by B-spline property (x), for modest order k, this control polygon cannot be too far from the curve itself. Sticking with a spline *function*, i.e., our scalar-valued spline $f = \sum_j a_j B_{jk}$, one infers directly from the dual functionals that

$$a_j = f(t^*_{jk}) + O((t_{j+k-1} - t_{j+1})^2 \|D^2 f\|_{[t_{j+1}..t_{j+k-1}]}).$$

This implies that the control polygon is close to the spline itself when the **mesh spacing**

$$|\mathbf{t}| := \sup_i (t_{i+1} - t_i)$$

is sufficiently small.

E.g., try out this simple example, in which a cubic spline is generated by interpolation, then plotted, along with its control polygon:

```
x = sort(rand(1,21))*4*pi; k = 4; sp = spapi(k,x,sin(x)./(.3+x));
fnplt(sp)
hold on, plot(aveknt(fnbrk(sp,'knots'),k), fnbrk(sp,'coef') ,'k'), hold off
```

If the mesh spacing isn't small enough, we can make it smaller simply by inserting more knots. After all, if $\mathbf{t}$ is a subsequence of $\widehat{\mathbf{t}}$, then $\$_{k,\mathbf{t}}$ is a subset of $\$_{k,\widehat{\mathbf{t}}}$, i.e.,

$$\mathbf{t} \subset \widehat{\mathbf{t}} \quad \Longrightarrow \quad \$_{k,\mathbf{t}} \subset \$_{k,\widehat{\mathbf{t}}},$$

hence, in that case, each $f \in \$_{k,\mathbf{t}}$ is also uniquely writeable as a weighted sum of the $\widehat{B}_{jk} := B_{j,k,\widehat{\mathbf{t}}}$:

$$\sum_j a_j B_{jk} = f = \sum_j \widehat{a}_j \widehat{B}_{jk}. \tag{1}$$

E.g., continue the example:

```
sp = fnrfn(sp,aveknt(x,3));
hold on, plot(aveknt(fnbrk(sp,'knots'),k), fnbrk(sp,'coef') ,'r'), hold off
```

The formula for the $\widehat{a}_j$ can be quite involved. However, we can obtain any **refinement** $\widehat{\mathbf{t}}$ of $\mathbf{t}$ in a sequence of steps, each of which consists of adding just *one* knot. Hence, it is sufficient to know the formula for $\widehat{a}_j$ in the special case that $\widehat{\mathbf{t}}$ is obtained from $\mathbf{t}$ by the insertion of just one additional knot. This formula constitutes B-spline property (xi). *If $\widehat{\mathbf{t}}$ is obtained from $\mathbf{t}$ by insertion of the point x, then (1) holds with*

$$\widehat{a}_j = \widehat{\omega}_{jk}(x) a_j + (1 - \widehat{\omega}_{jk}(x)) a_{j-1},$$

where

$$\widehat{\omega}_{jk}(x) := \max\{0, \min\{1, \omega_{jk}(x)\}\}.$$

