cs 514, lecture 7apr00 – ??:recap re data maps, column maps, recovery, projectors, ...

Recall the setup so far. We are hoping to recover an element f of the ls F from information $\Lambda' f = (\lambda_i f : i \in I)$ about it, with each λ_i a lfnl, hence

$$\Lambda' : F \to \mathbb{F}^I$$

a linear map. We assume that Λ' is 1-1.

We only looked at the finite-dimensional case:

$$\Lambda' f = (\lambda_i : i = 1, \ldots, n),$$

and assumed that Λ' is onto.

Then F is necessarily n-dimensional, hence has a basis consisting of n elements, or, equivalently, there is a reconstruction map, or column map,

$$W : \mathbb{F}^n \to F : a \mapsto \sum_{j=1}^{n} w_j a(j) =: [w_1, \ldots, w_n]a$$

that is invertible.

Under that assumption, the Gram matrix $\Lambda' W = (\lambda_i w_j : i, j = 1, \ldots, n)$ is invertible, and the formula

$$f = W(\Lambda' W)^{-1} \Lambda' f, \qquad \forall f \in F,$$

provides the complete reconstruction of $f \in F$ from the information $\Lambda' f$ about it.

We observed that this amounts to nothing more than a change of basis: Since $\Lambda' : F \to \mathbb{F}^n$ is invertible, its inverse, $V := (\Lambda')^{-1}$, is a linear map, hence necessarily of the form

$$V : \mathbb{F}^n \to F : a \mapsto \sum_{j=1}^{n} v_j a(j) =: [v_1, \ldots, v_n]a,$$

and, necessarily,

$$\mathrm{id} = \Lambda' V = (\lambda_i v_j : i, j = 1, \ldots, n)$$

the identity matrix, i.e., the two sequences (λ_i) and (v_j) are biorthonormal and, correspondingly,

$$f = V \Lambda' f = \sum_{j=1}^{n} v_j \lambda_j f, \qquad \forall f \in F.$$

In fact, necessarily

$$V = W(\Lambda' W)^{-1}.$$

Hence, our reconstruction consisted of the following: obtain, from the coordinates $\Lambda' f$ of f wrto the basis V, the coordinates $(\Lambda' W)^{-1} \Lambda' f$ of f wrto the basis W.

Nevertheless, we counted this a success to the extent that the coordinates of f wrto the basis W, more readily than $\Lambda' f$, provide the information about f we are after.

This led us to the basic question: *What information about $f \in F$ is readily obtainable from its coordinates wrto a given basis W for F?*

We are going to look in detail into this question for the case that F is a spline space, and the basis in question is its B-spline basis.

But before starting that discussion, here are two more general observations at this more abstract level, that will be important.

1. **linear projectors**: Extend our setup slightly, by assuming that our linear space F is actually a linear subspace of some linear space X. A concrete example would be

$$F = \Pi_{<n} \subset C[0\,..\,1] =: X.$$

Correspondingly, assume that our data map, Λ', is actually defined on all of X. For our earlier concrete example,

$$\Lambda' f = (f(\tau_i) : i = 1, \ldots, n),$$

this is obviously true (provided $\tau_i \in [0\,..\,1]$, all i). Then we can construct

$$Pg := W(\Lambda' W)^{-1} \Lambda' g$$

for every $g \in X$, but, while $g = Pg$ whenever $g \in F$, this need not hold for every $g \in X$. In fact, since necessarily $Pg \in F$ (why?), it follows that

$$g = Pg \quad \iff \quad g \in F.$$

In other words,

$$\operatorname{ran} P = F, \quad PP = P.$$

A linear map with the property $P^2 = P$ is called **idempotent** or a **projector**. So, P is a linear projector onto F.

What is the relationship between $g \in X$ and Pg? Since

$$\Lambda' Pg = \Lambda' W(\Lambda' W)^{-1} \Lambda' g = \Lambda' g,$$

we recognize Pg as the *unique element of F that agrees with g on Λ'* and therefore call it the unique **interpolant from F to the data $\Lambda' g$**. The most commonly used approximation schemes are all of this form, with the only difference being the choice of the data map, i.e., the choice of the information to be matched.

By considering g from a superspace for F, we cannot hope any more for lossless reconstruction; still, it is reconstruction of a sort.

2. In the first part of this course, a situation more special and more general was considered. The ls F was usually infinite-dimensional, but a Hilbert space. But, even when when F was finite-dimensional, the data map, while 1-1, was not assumed to be onto. This provided some extra flexibility, but it is important to realize what was lost.

If $\Lambda' = [\lambda_1, \ldots, \lambda_n]' : F \to \mathbb{F}^n$ is not onto, then

$$m := \dim F < n.$$

In particular, our basis W for F now is a map from $\mathbb{F}^m$ rather $\mathbb{F}^n$, hence, offhand, not helpful for the reconstruction. Rather, we now must choose some column map $U : \mathbb{F}^n \to F$, necessarily not 1-1, so that

$$f = U\Lambda' f, \quad \forall f \in F,$$

but it is not obvious how to do this. If U is such a map, then, certainly, the Gramian $\Lambda' U$ must be the identity on $\operatorname{ran} \Lambda'$, but that is far from having it be the identity matrix.

In contrast, if all we know is that Λ' is some data map to $\mathbb{F}^n$ and that $F = \operatorname{ran} W$ for some reconstruction map $W : \mathbb{F}^n \to X$, then just the fact that the Gramian $\Lambda' W$, which is square, is either 1-1 or onto tells us already that it must be invertible, and Λ' must be onto even when restricted to F, and W must be a basis for F, and we get the earlier reconstruction formula for all $f \in F$.

A standard sufficient condition for a square matrix to be invertible is for the matrix to be triangular with nonzero diagonal entries.

