(1) B-spline Property (i): Recurrence relation. *For $k > 1$,*

$$\mathrm{B}_{jk} = \omega_{jk}\mathrm{B}_{j,k-1} + (1 - \omega_{j+1,k})\mathrm{B}_{j+1,k-1}, \tag{2}$$

with

$$\omega_{jk}(x) := \begin{cases} \dfrac{x - t_j}{t_{j+k-1} - t_j}, & \textit{if } t_j < t_{j+k-1}; \\ 0, & \textit{otherwise.} \end{cases} \tag{3}$$

(4) B-spline Property (ii): Support and positivity. *The B-spline $\mathrm{B}_{j,k,\mathbf{t}}$ is pp of order k with breaks $t_j, \ldots, t_{j+k}$, hence consisting of at most k nontrivial polynomial pieces, vanishes outside the interval $[t_j \,..\, t_{j+k})$, and is positive on the interior of that interval, that is,*

$$\mathrm{B}_{j,k,\mathbf{t}}(x) > 0, \quad t_j < x < t_{j+k} \tag{5}$$

while

$$t_j = t_{j+k} \implies \mathrm{B}_{jk} = 0. \tag{6}$$

(7) B-spline Property (iii): Marsden's Identity. *For any $\tau \in \mathbb{R}$,*

$$(\cdot - \tau)^{k-1} = \sum_j \psi_{jk}(\tau)\mathrm{B}_{jk} \tag{8}$$

with

$$\psi_{jk}(\tau) := (t_{j+1} - \tau) \cdots (t_{j+k-1} - \tau). \tag{9}$$

(10) B-spline Property (iv): (positive and local) partition of unity. *The sequence (B_{jk}) provides a positive and local partition of unity, that is, each B_{jk} is positive on $(t_j \,..\, t_{j+k})$, is zero off $[t_j \,..\, t_{j+k}]$, and*

$$\sum_j \mathrm{B}_{jk} = 1 \qquad \textit{on } \; I_{k,\mathbf{t}}, \tag{11}$$

with $I_{k,\mathbf{t}} = (\mathbf{t}_- \,..\, \mathbf{t}_+)$ the basic interval for

$$\$_=\$_{k,\mathbf{t}} := \mathrm{span}(\mathrm{B}_{jk} : j),$$

i.e.,

$$\mathbf{t}_- := \begin{cases} t_k, & \textit{if } \mathbf{t} = (t_1, \ldots); \\ \inf_j t_j, & \textit{otherwise,} \end{cases} \qquad \mathbf{t}_+ := \begin{cases} t_{n+1}, & \textit{if } \mathbf{t} = (\ldots, t_{n+k}); \\ \sup_j t_j, & \textit{otherwise.} \end{cases}$$

(12) B-spline Property (v): Knot averages. *For $k > 1$ and any $\ell \in \Pi_{<2}$,*

$$\ell = \sum_j \ell(t^*_{jk})\ \mathrm{B}_{jk}$$

with t^*_{jk} the **Greville sites***:*

$$t^*_{jk} := \frac{t_{j+1} + \cdots + t_{j+k-1}}{k-1}, \quad \text{all } j. \tag{13}$$

(14) B-spline Property (vi): Local linear independence. *For any knot sequence $\mathbf{t}$, and any interval $I = [a\,..\,b] \subseteq I_{k,\mathbf{t}}$ containing finitely many of the t_i, the sequence*

$$B := (\mathrm{B}_{j,k,\mathbf{t}|I} : \mathrm{B}_{j,k,\mathbf{t}|I} \neq 0)$$

is a basis for $\Pi_{<k,\boldsymbol{\xi},\boldsymbol{\nu}|I}$, with $\boldsymbol{\xi}$ the strictly increasing sequence containing a, b, as well as every $t_i \in I$, and $\nu_i := k - \min(k, \#\{r : t_r = \xi_i\})$. In particular, B is linearly independent.

(15) B-spline Property (vii): Dual functionals. (de Boor & Fix [1973]) *For any $f \in \$_{k,\mathbf{t}}$,*

$$f \;=\; \sum_j \lambda_{jk} f\ \mathrm{B}_{jk},$$

with

$$\lambda_{jk} f := \sum_{\nu=1}^{k} \frac{(-D)^{k-\nu}\psi_{jk}(\tau_j)}{(k-1)!}\ D^{\nu-1} f(\tau_j) \tag{16}$$

and $t^+_j \le \tau_j \le t^-_{j+k}$, all j. Hence

$$\lambda_{ik}\Big(\sum_j \alpha_j\ \mathrm{B}_j\Big) \;=\; \alpha_i, \quad \text{all } i. \tag{17}$$

(18) B-spline Property (viii): Differentiation.

$$D\left(\sum_j \alpha_j \mathrm{B}_{jk}\right) = (k-1) \sum_j \frac{\alpha_j - \alpha_{j-1}}{t_{j+k-1} - t_j}\,\mathrm{B}_{j,k-1}. \tag{19}$$

(20) B-spline Property (ix): Convex hull. *For $t_i < x < t_{i+1}$, the value of the spline function $f := \sum_j \alpha_j \mathrm{B}_j$ at the site x is a strictly convex combination of the k numbers $\alpha_{i+1-k}, \ldots, \alpha_i$.*

(21) B-spline Property (x): Good condition. $(\mathrm{B}_i)_1^n$ *is a relatively well conditioned basis for* \$ *in the sense that there exists a positive constant* $D_{k,\infty}$*, which depends only on* k *and not on the particular knot sequence* $\mathbf{t}$*, so that for all* i*,*

$$|\alpha_i| \le D_{k,\infty} \Big\| \sum_j \alpha_j \mathrm{B}_j \Big\|_{[t_{i+1} \,.\,. t_{i+k-1}]}. \tag{22}$$

Smallest possible values for $D_{k,\infty}$ *are*

k	2	3	4	5	6
$D_{k,\infty}$	1	3	$5.5680\cdots$	$12.0886\cdots$	$22.7869\cdots$

Based on numerical calculations, it is conjectured that, in general,

$$D_{k,\infty} \sim 2^{k-3/2}.$$

(23) B-spline Property (xi): Knot insertion. *(*W. Boehm [1980]) *If the knot sequence* $\widehat{\mathbf{t}}$ *is obtained from the knot sequence* $\mathbf{t}$ *by the insertion of just one term,* x *say, then, for any* $f \in \$$, $\sum_j \alpha_j \mathrm{B}_{k,\mathbf{t}} := f =: \sum_j \widehat{\alpha}_j \mathrm{B}_{k,\widehat{\mathbf{t}}}$ *with*

$$\widehat{\alpha}_j = (1 - \widehat{\omega}_{jk}(x))\alpha_{j-1} + \widehat{\omega}_{jk}(x)\alpha_j, \quad \text{all } j, \tag{24}$$

and $\widehat{\omega}_{jk} := \max\{0, \min\{1, \omega_{jk}\}\}$*, i.e.,*

$$\widehat{\omega}_{jk} : x \mapsto \begin{cases} 0, & \text{for } x \le t_j; \\ \omega_{jk}(x) = \dfrac{x - t_j}{t_{j+k-1} - t_j}, & \text{for } t_j < x < t_{j+k-1}; \\ 1, & \text{for } t_{j+k-1} \le x. \end{cases} \tag{25}$$

(26) B-spline Property (xii): Variation diminution. *If* $f = \sum_j \alpha_j \mathrm{B}_j$ *and* $\tau_1 < \cdots < \tau_r$ *are such that* $f(\tau_{i-1})f(\tau_i) < 0$*, all* i*, then one can find indices* $1 \le j_1 < \cdots < j_r \le n$ *so that*

$$\alpha_{j_i} \mathrm{B}_{j_i}(\tau_i) f(\tau_i) > 0 \qquad \text{for } i = 1, \ldots, r. \tag{27}$$