Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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Prelude

Given a surjective linear map

$$X : \mathbb{R}^m \to \mathbb{R}^n,$$

the **zonotope** $Z(X)$ is the image (in $\mathbb{R}^n$) of the m-dimensional unit cube; thus, considering X as a multiset of vectors in $\mathbb{R}^n$,

$$Z(X) = \left\{ \sum_{x \in X} t_x x, \quad t_x \in [0,1],\ \forall x \in X \right\}.$$

By duality, the multiset X can also be considered as a collection of linear functionals:

$$p_x : \mathbb{R}^n \to \mathbb{R} \ : \ t \mapsto x \cdot t := \sum_{i=1}^{n} x(i)t(i), \quad x \in X.$$

Let H_x by the zero-set of p_x:

$$H_x := \{t \in \mathbb{R}^n : \ p_x(t) = 0\}.$$

The multiset of hyperplanes

$$\mathcal{H}(X) := \{H_x : \ x \in X\},$$

is known as a **(central) hyperplane arrangement**, and, as one should expect, records the geometry of the zonotope $Z(X)$ in a dual (i.e., polar) way. The geometric duality between the zonotope $Z(X)$ and the arrangement $\mathcal{H}(X)$ is well-known, and will only be briefly touched in these notes.

My personal interest in zonotopes stems from their role in Approximation Theory: the zonotope $Z(X)$ is the support of a certain smooth piecewise-polynomial, and, more generally, a class of piecewise-exponentials known as *(exponential) box splines*. Approximation Theory deals with the properties of the shift-invariant function space that is spanned by the integer translates of one or more box splines. Such a space is known as a *box spline space*. For all practical purposes, we assume that X is integral:

$$X \subset \mathbb{Z}^n.$$

Moreover, we frequently assume X to be *unimodular* (a notion that is defined later), since in this case the shifts of the box spline are known to be *linearly independent*.

Zonotopes in general, and unimodular zonotopes in particular, are of interest outside Approximation Theory, too. For example, a standard representation of the edge set $E(G)$ of a connected graph with $n+1$ vertices is with the aid a special type of unimodular X. Clearly, graph theorists may not be interested (at least not initially) in the existence of smooth functions supported on $Z(X)$. They may be (so I hope) interested in certain combinatorial properties of zonotopes, and the relations of these combinatorics to the spanning trees, the forests, and other basics of the underlying graph. If *box spline* is the buzzword that relates the zonotope to Approximation Theory, then the *Tutte polynomial* is (perhaps) the counterpart in the context of Graph Theory.

We introduced so far the zonotope as a geometric object. We hinted about a related analytic setup (box splines), a related combinatorial (as well as Linear Algebra) setup (via graphs), and made an initial mentioning of *geometric duality*. None of these is a core focus of these notes. In particular, we move away from the Linear Algebra setup, where the linear combinations of the linear functionals p_x are at the center, and view instead each p_x as a polynomial in the polynomial ring

$$\Pi := \mathbb{C}[t_1, \ldots, t_n]$$

of n-variables. Thus, we consider products of the form

$$p_Y := \prod_{x \in Y} p_x, \quad Y \subset X.$$

Given a subset $S \subset 2^X$ of subsets of X, we have two complementary ways for viewing the polynomials p_Y, $Y \in S$:

A $\mathcal{P}$**-type space** is defined as

$$\mathcal{P}_S := \text{span}\{p_Y : Y \in S\}.$$

In contrast, a $\mathcal{J}$**-type ideal** is defined as

$$\mathcal{J}_L := \text{Ideal}\{p_Y : Y \in L\},$$

with $L \subset 2^X$, too. To be sure, given subset $A \subset \Pi$, the notation

$$\text{Ideal}(A) := \text{Ideal}\{p : p \in A\}$$

stands for the smallest ideal in Π that contains A, we may sometimes denote as

$$\Pi A.$$

We always assume (without loss) that

$$(Y \in L,\ Y \subset Y' \subset X) \Longrightarrow Y' \in L,$$

and conversely (with loss) that

$$(Y \in S,\ Y \supset Y') \Longrightarrow Y' \in S.$$

Ideals of the type $\mathcal{J}_L$ play a role in box spline theory. They also have strong (albeit somewhat implicit) presence in Lie group representations. Ideally, we look for subsets $S, L \subset 2^X$ such that

$$\Pi = \mathcal{P}_S \oplus \mathcal{J}_L, \tag{1}$$

which can be correctly (and later on, rigorously) regarded as a form of algebraic duality. It is clear that (1) holds only if S and L are disjoint. We will present in these notes three different setups (each of which applies to a general X) where the above split of Π holds, and refer to these as *central, external* and *internal* respectively. In each of these setups, we will show that the space $\mathcal{P}_S$ is related in suitable way to the zonotope $Z(X)$, while the ideal $\mathcal{J}_L$ is connected with the hyperplane arrangement $\mathcal{H}(X)$.

As far as the history of this theory is concerned, the central part, which will occupy most of our lectures, was developed within Approximation Theory during the period 1983-1990. My interest in the external and internal parts dates back to 1996, which was when the pertinent definitions were made. However, some of the main sought-for results were proved only recently, and, to the best of my knowledge, the material concerning the internal and external theory appears here for the first time.[1]

[1]This last statement was true at the time these lectures were given at the University of Wisconsin (January 2007). A complete account of the external and internal theories appears now in the paper *Zonotopal Algebra*, joint with Olga Holtz, that can be fetched at `ftp://ftp.cs.wisc.edu/Approx/zonotopes.pdf` .