Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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The topic will bring together aspects of linear algebra, convex geometry, combinatorics, commutative algebra, and analysis.

1 Linear Algebra.

Our entire investigation concerns a finite multiset $X \subset \mathbb{R}^n \backslash \{0\}$ of full rank n. At times, we will associate X with some (full) ordering. In this case, we may consider the vectors in X to comprise the columns of an $n \times X$ matrix, which we will still denote by X.

The set of all bases of X is denoted by $\mathbb{B}(X)$:

$$\mathbb{B}(X) := \{ B \subset X : B \text{ is a basis for } \mathbb{R}^n \}.$$

Example 1.1 Let X be the following multiset of four vectors $x_1, \ldots, x_4$ in $\mathbb{R}^2 \backslash \{0\}$:

$$X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = [x_1, x_2, x_3, x_4].$$

Note that we have here 5 bases. To visualize a basis $B \in \mathbb{B}(X)$, we consider B as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and choose the parallelepiped $B([0,1]^n)$ as our visualization of B. In this example we have the following 5 bases:

$$[x_1, x_3],\ [x_2, x_3],\ [x_1, x_4],\ [x_2, x_4],\ [x_3, x_4],$$

and they correspond to the following 5 parallelepipeds:

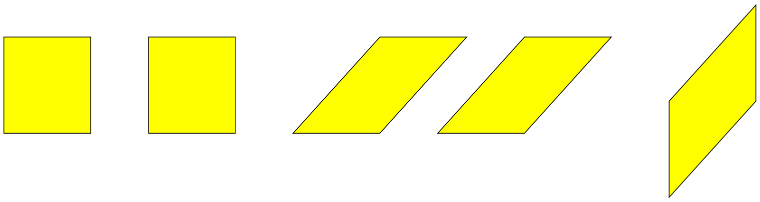

Figure 1: The five parallelograms of Example 1.1

□

We next define

$$\mathbb{I}(X) := \{\, I \subset X,\ I \text{ is independent in } \mathbb{R}^n \,\}\,.$$

Note that the empty set is defined to be independent. Thus, in Example 1.1, we have $\#\mathbb{I}(X) = 10$.

It is sometimes useful to extend the independent sets to full-rank bases. We do this extension *externally*: we choose some fixed basis B_0 of $\mathbb{R}^n$, and append B_0 to X:

$$X' := X \cup B_0.$$

We then impose some arbitrary, but fixed, ordering on B_0, and associate each $I \in \mathbb{I}(X)$ with $\mathrm{ex}(I) := B_I \in \mathbb{B}(X')$ which is the greedy completion of I to a basis, using the elements of B_0; i.e.,

$$\mathrm{ex}(I) := I \cup \{b \in B_0 :\ b \notin \mathrm{span}\{I \cup \{b' \in B_0 : b' < b\}\}.$$

This defines a 1-1 map $\mathrm{ex} : \mathbb{I}(X) \to \mathbb{B}(X')$. We will denote by

$$\mathbb{B}_+(X) \subset \mathbb{B}(X')$$

the range of this map.

We refer to the bases in $\mathbb{B}_+(X)$ as the **external bases** of X. Note that every basis of X is external, by the definition of external bases.

Next, we define the notion of an *internal basis*. To this end, we order X in an arbitrary way.[1] A vector $b \in B \in \mathbb{B}(X)$ is said to be **internally active** if b is the maximal (i.e., last) element in $X\backslash H$, where $H := \mathrm{span}(B\backslash b)$. The number of vectors in $B \in \mathbb{B}(X)$ that are *not* internally active is denoted by

$$\mathrm{val}^*(B).$$

A basis $B \in \mathbb{B}(X)$ that has no internally active vectors is an **internal basis**. I.e., B is internal iff $\mathrm{val}^*(B) = n$. We denote

$$\mathbb{B}_-(X) := \{\, B \in \mathbb{B}(X)\ :\ B \text{ is an internal basis }\,\}\,.$$

Example 1.2 Suppose X is ordered in the way given by Example 1. Then

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \notin \mathbb{B}_-(X), \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{B}_-(X).$$

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[1] In fact, assuming that the last n vectors in that order form a basis $B_1 \in \mathbb{B}(X)$, only the internal order within the basis B_1 is important here. This point of view reveals the duality between the two definitions of external bases and internal ones.

In fact,

$$\mathbb{B}_-\left(\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\right) = \{\, [x_1, x_3]\,,\ [x_2, x_3]\,\}\,.$$

□

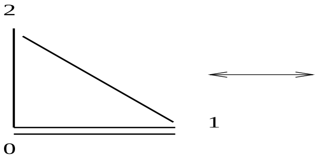
It is obvious that the notion of an internal basis depends on the ordering. It is known that the number of internal bases is independent of the order of X. In fact, the number of internal bases is completely determined by the independent set $\mathbb{I}(X)$ as follows:

$$\#\mathbb{B}_-(X) = (-)^n \sum_{I \in \mathbb{I}(X)} (-)^{\#I}.$$

Definition 1.3 *We say that* X *is* **unimodular** *if* $\forall B \in \mathbb{B}(X)$, $\mathrm{span}_{\mathbb{Z}} B = \mathbb{Z}^n$ *(equivalently,* $B \subset \mathbb{Z}^n$ *and* $\det(B) = \pm 1$.

Note the a unimodular X is necessarily integral. Note also that multiset X in Example

Example 1.4 *The edge set of a graph* G*.*
Let G *be a connected undirectional graph with* $n+1$ *vertices. Let* $e_0 := 0$*. Let* $(e_i)_{i=1}^n$ *be the standard basis for* $\mathbb{R}^n$*. An edge* e_{ij} *that connects the vertices* i *and* j *is associated with the vector* $e_i - e_j \in \mathbb{R}^n$*. We then choose* X *to be the "edge set" of* G*. For example, the graph below is associated with the multiset to its right:*



$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Figure 2: An edge set X

□

Note that the edge (multi)set X of a graph is always unimodular.

2 Hyperplane arrangements

We first associate each $x \in X$ with a constant $\lambda_x \in \mathbb{R}$, and define a linear polynomial $p_x := p_{x,\lambda}$:

$$p_x : \mathbb{R}^n \to \mathbb{R} : t \mapsto x \cdot t - \lambda_x.$$

The hyperplane arrangement $\mathcal{H}(X,\lambda)$ of X is the arrangement induced by the zero sets of the above polynomials, viz., by

$$H_{x,\lambda} := \{\, t \in \mathbb{R}^n \,:\, p_x(t) = 0 \,\}, \quad x \in X.$$

We will assume that (λ_x) are chosen so that the intersection of any collection of $n+1$ hyperplanes is empty. We refer to this case as **generic**. Note that different (generic) choices of (λ_x) result possibly in hyperplane arrangements with different geometries. For example, in Example 1, we have the following 3 hyperplane arrangements with (at least) two different geometries.

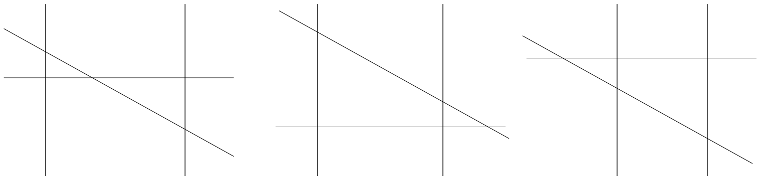


Figure 3: Hyperplane arrangements with different geometries

We will focus on three geometrical objects in the hyperplane arrangement.

1. $V(X,\lambda)$: the set of vertices

2. $CC(X,\lambda)$: the set of n-dimensional connected components

3. $BCC(X,\lambda)$: the set of n-dimensional bounded connected components

We claim that the following relations hold:

$$\#V(X,\lambda) = \#\mathbb{B}(X), \quad \#CC(X,\lambda) = \#\mathbb{I}(X), \quad \#BCC(X,\lambda) = \#\mathbb{B}_-(X).$$

The first relation is straightforward: there is a natural surjection

$$\mathbb{B}(X) \to V(X,\lambda) \;:\; B \mapsto v_B := \cap_{b\in B} H_{b,\lambda},$$

and this map is injective in the generic case.

Finally, let $\mathbb{I}_k(X)$ be the collection of independent sets of X whose cardinality is k. Then

$$\mathbb{I}(X) = \dot\cup_{k=0}^{n} \mathbb{I}_k(X).$$

Now, define a polynomial:

$$P_X(t) := \sum_{k=0}^{n} (\#\mathbb{I}_k(X))\, t^{n-k}, \quad t \in \mathbb{R}.$$

Then $P_X(1) = \#\mathbb{I}(X)$, $\quad P_X(-1) = \#\mathbb{B}_-(X)$. The first statement here is trivial. The other one is not hard, either.

3 Zonotopes: the dual geometry of hyperplane arrangements

Now, let's consider X as a map:

$$X : \mathbb{R}^X \to \mathbb{R}^n : t \mapsto \sum_{x \in X} t_x x.$$

Then the zonotope of X is defined by

$$Z(X) := \mathrm{Im}(X|_{[0,1]^X}).$$

We denote

$$\mathcal{Z}_+(X) := Z(X) \cap \mathbb{Z}^n,$$

and denote by

$$\mathcal{Z}_-(X)$$

the subset of $\mathcal{Z}_+(X)$ that lies in the interior of $Z(X)$. Assuming X to be unimodular, we have that

1. $\mathrm{vol}(Z(X)) = \#\mathbb{B}(X)$,

2. $\#\mathcal{Z}_+(X) = \#\mathbb{I}(X)$,

3. $\#\mathcal{Z}_-(X) = \#\mathbb{B}_-(X)$.

Example 3.1 Let

$$X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =: [x_1 \ x_2 \ x_3 \ x_4].$$

In order to find $Z(X)$, we start at origin and move in the direction of x_1, x_2, x_4, and x_3 (in this order). We then go back, using the vectors $-x_1$, $-x_2$, $-x_4$, $-x_3$, in the same order as before.

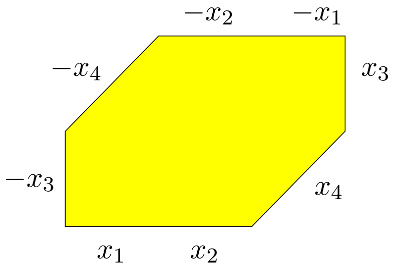


Figure 4: The zonotope $Z(X)$ of Example 1.

Every zonotope $Z(X)$ is the essentially disjoint union of the translated parallelepipeds

$$t_B + Z(B), \quad B \in \mathbb{B}(X).$$

The translation $t_B \in \mathbb{R}^n$ satisfies $t_B = \sum_{x \in Y_B}$, for a suitable $Y_B \subset X \backslash B$. Of course, the above description does not determine t_B (one is yet to choose Y_B...), and there are multiple ways to choose the translations (t_B); each such complete choice yields a tiling of the zonotope. Three such tilings of the above zonotope are displayed below. Each tiling corresponds to an ordering of X, and each such ordering corresponds to a different geometry on the hyperplane arrangement. In this duality, the vertices of the hyperplane arrangements are associated with the parallelepipeds that tile the zonotope, the bounded regions of the arrangement correspond to the interior lattice points in the zonotope, and the unbounded regions of the arrangement correspond to the lattice points on the boundary of the zonotope.[2] Thus, for example, the number of vertices that belong to a connected region of the arrangement must agree with the number of parallelepipeds that contain a given lattice point of the zonotope in their closure.

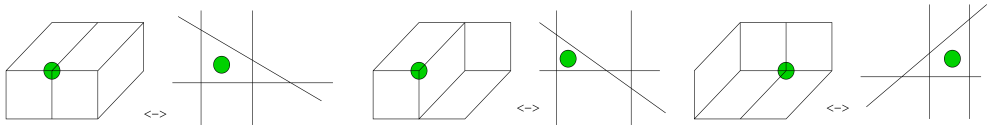


Figure 5: Tiling and hyperplane arrangements

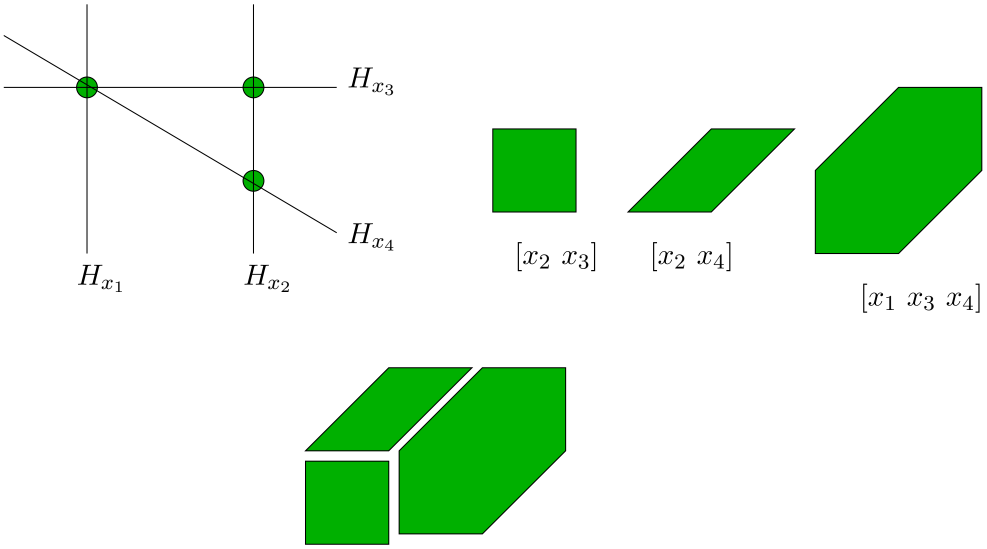


Figure 6: Tiling with 2 parallelograms and 1 smaller zonotope

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[2] If X is not unimodular, one needs to replace "lattice points" by the vertices of the parallelepipeds that tile the zonotope.

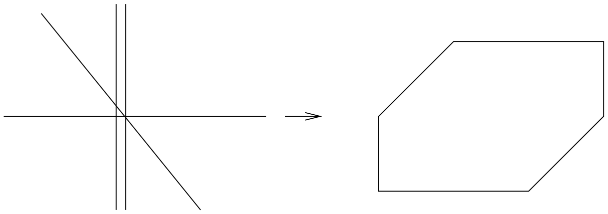

Figure 7: The central arrangement corresponds to no tiling