

Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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1 External Theory continued

We want to analyse further the space $\mathcal{P}_+(X)$. Note that Corollary 1.1 of Lecture 9 asserts that $\dim \mathcal{P}_+(X) \geq \#\mathbb{I}(X)$. We will next construct a basis for $\mathcal{P}_+(X)$. As in the case of $\mathcal{P}(X)$, there are two (completely) different approaches for constructing such a basis.

1. *Inhomogeneous* bases for $\mathcal{P}_+(X)$:

We pick a generic¹ $\lambda \in \mathbb{R}^X$, and associate, as usual, each $x \in X$ with an inhomogeneous linear polynomial $p_x : t \mapsto x \cdot t - \lambda_x$. Then

$$\mathcal{B}_+^\lambda := \{ p_{X \setminus I} : I \in \mathbb{I}(X) \}$$

is a basis for $\mathcal{P}_+(X)$.

The fact that each $p_{X \setminus I}$ lies in $\mathcal{P}_+(X)$ is more or less obvious. It is not hard to prove that the polynomials in \mathcal{B}_+^λ are independent (the easiest way to prove it is to show that they are linearly independent on the vertex set $V(X', \lambda')$ of the extended hyperplane arrangement (with X' as below, and λ' a generic extension of λ)). The fact that \mathcal{B}_+^λ spans $\mathcal{P}_+(X)$ can be proved in two steps: in the first step one shows that every polynomial of the form p_Y , $Y \subset X$, lies in $\text{span} \mathcal{B}_+^\lambda$, and in the second step one shows that every polynomial q_Y , $Y \subset X$ also lies in \mathcal{B}_+^λ . The proof altogether is quite simple, and is omitted only because we will not need this type of construction for the core development of the external theory.

2. *Homogeneous* bases for $\mathcal{P}_+(X)$:

In this case, we order X , and define, for each $I \in \mathbb{I}(X)$,

$$Y_I := \{ x \in X : x \notin \text{span} \{ b \in I : b \leq x \} \}.$$

¹Recall that λ is considered generic if the association between $\mathbb{B}(X)$ and the vertex set $V(X, \lambda)$ of the hyperplane arrangement is 1-1.

Then²

$$\mathcal{B}_+^0 := \{ Q_I := q_{Y_I} : I \in \mathbb{I}(X) \}$$

is a basis for $\mathcal{P}_+(X)$.

Note that the basis \mathcal{B}_+^0 is an extension of the homogeneous basis that was constructed for $\mathcal{P}(X)$. Moreover, the valuation val that was defined there on $\mathbb{B}(X)$, has just been extended in the most natural way to $\mathbb{I}(X)$:

$$\text{val}(I) := \#Y_I, \quad I \in \mathbb{I}(X).$$

This motivates us to associate X with *an external Hilbert function*:

$$h_+ := h_{X,+} : \mathbb{N} \rightarrow \mathbb{N} \quad : \quad j \mapsto \#\{I \in \mathbb{I}(X) : \text{val}(I) = j\}.$$

$h_+(j)$ equals thus to the dimension of $\mathcal{P}_+(X) \cap \Pi_j^0$.

The *proof* that \mathcal{B}_+^0 is a basis for $\mathcal{P}_+(X)$ is done in two steps: in the first we show that the polynomials Q_I , $I \in \mathbb{I}(X)$ are linearly independent. For the proof of this part, we order the augmented set $X' := X \cup B_0$ such that X retains its internal order, and B_0 is placed after X . Recall that each $I \in \mathbb{I}(X)$ has a well-defined extension to a basis $\text{ex}(I) \in \mathbb{B}(X')$:

$$\mathbb{I}(X) \rightarrow \mathbb{B}_+(X) \subset \mathbb{B}(X') : I \mapsto \text{ex}(I).$$

We therefore examine the known homogeneous basis for $\mathcal{P}(X')$. The polynomials in that latter basis are $q_{Y'_B}$, $B \in \mathbb{B}(X')$, with

$$Y'_B := \{x \in X' : x \notin \text{span}\{b \in B : b \leq x\}\}.$$

Now, given $I \in \mathbb{I}(X)$, since $\text{ex}(I) \in \mathbb{B}(X')$ is a *greedy* extension of I , it easily follows that

$$Y_I = Y'_{\text{ex}(I)},$$

hence that

$$Q_I := q_{Y_I} = q_{Y'_{\text{ex}(I)}}.$$

Thus, the set \mathcal{B}_+^0 is a subset of the basis $\{q_{Y'_B} : B \in \mathbb{B}(X')\}$ for $\mathcal{P}(X')$. Therefore the polynomials in \mathcal{B}_+^0 are linearly independent

The remaining step is to prove the following:

Lemma 1.1 *Define*

$$I_+(X) := \text{Ideal} \left\{ q_{\eta^H}^{m(H)+1} : H \in \mathcal{H}^*(X) \right\}.$$

Then

$$\dim \ker I_+(X) \leq \#\mathbb{I}(X).$$

²The notation below cannot be confused with the notation \mathcal{B}_+^λ , since the choice $\lambda = 0$ is not generic.

We will sketch the proof of the lemma at the end of this section. For the time being, we combine the lemma with the following facts:

- $\mathcal{P}_+(X) \subset \ker I_+(X)$ (which is easy to check)
- $\mathcal{D}_+(X) \leftrightarrow \mathcal{P}_+(X)'$ (which we proved in Lecture 9)
- $\dim \mathcal{D}_+(X) \geq \#I(X)$ (which follows from Lecture 2, as discussed in Lecture 9)

to yield most of the claims in the following theorem:

Theorem 1.2

- (1) $\dim \mathcal{P}_+(X) = \dim \mathcal{D}_+(X) = \#I(X)$.
- (2) The map $p \mapsto \langle p, \cdot \rangle$ is a bijection between $\mathcal{D}_+(X)$ and $\mathcal{P}_+(X)'$.
- (3) $\mathcal{D}_+(X) = \ker J_+(X)$.
- (4) $\mathcal{P}_+(X) = \ker I_+(X)$.
- (5) The polynomials $Q_I := q_{Y_I}$, $I \in I(X)$, form a basis for $\mathcal{P}_+(X)$.
- (6) Let $V_+(X, \lambda)$ be the subset of vertex set $V(X', \lambda')$ (of the hyperplane arrangement (X', λ')) that corresponds to $\mathbb{B}_+(X) \subset \mathbb{B}(X')$. Assume λ' to be generic. Then

$$\Pi(V_+(X, \lambda)) = \mathcal{D}_+(X).$$

- (7) Let \mathcal{Z}_+ be the integer points in the closed zonotope $Z(X)$. Then

$$\Pi(\mathcal{Z}_+) = \mathcal{P}_+(X),$$

provided that X is unimodular.

The only statements in the theorem that do not follow directly from the discussion so far are the last two. For (6), we can follow almost *verbatim* the argument that shows that $\Pi(V(X, \lambda)) \subset \mathcal{D}(X)$ (Lecture 3) to conclude that

$$\Pi(V_+(X, \lambda)) \subset \mathcal{D}_+(X).$$

The assertion then follows from (1) and the fact that

$$\dim \Pi(V_+(X, \lambda)) = \#V_+(X, \lambda) = \#\mathbb{B}_+(X) = \#I(X).$$

For the proof of (7), we need to know first that

$$\#\mathcal{Z}_+ = \#I(X),$$

which is true for a unimodular X . That implies that

$$\dim \Pi(\mathcal{Z}_+) = \#\mathbb{I}(X) = \dim \mathcal{P}_+(X).$$

Hence (7) follows from the facts that $\mathcal{P}_+(X) = \ker I_+(X)$ and that

$$\Pi(\mathcal{Z}_+) \subset \ker I_+(X).$$

The proof of this latter inclusion requires us to show (cf. Lecture 2) that, given any generator $q := q_{\eta_H}^{m(H)+1}$, $H \in \mathcal{H}^*(X)$, of $I_+(X)$, there exists $p \in \Pi$ that vanishes on \mathcal{Z}_+ and satisfies $p_{\uparrow} = q$. The existence of such p follows from the fact that, whatever dual hyperplane H we choose, the set \mathcal{Z}_+ lies in the union

$$\cup_{j=0}^{m(H)} (a_j + H),$$

with $a_j := \sum_{k=1}^j x_k$, and where $\{x_j\}_{j=1}^{m(H)} = X \setminus H$ (the hyperplanes in the above union do not depend on the order we impose on $X \setminus H$. Also, the description above assume that $X \setminus H$ all lie on the same side of H ; the modifications that are needed for the general case are notational).

Here is an example.

Example 1.3

$$X' = X \cup B_0 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} =: [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6]$$

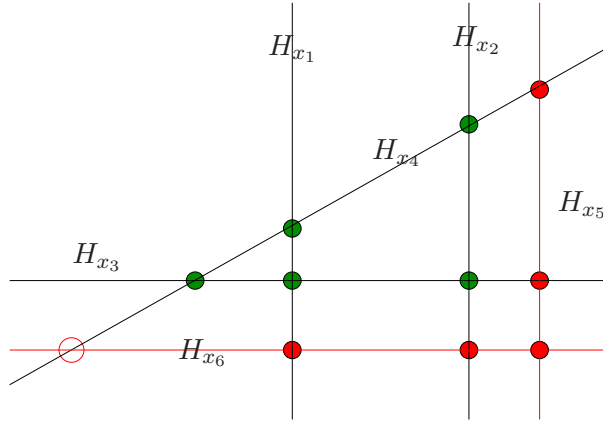


Figure 18. $V_{\mathbb{B}_+(X)}(X', \lambda)$

The green points correspond to the vertex set $V(X, \lambda)$ (for which we have the relation $\Pi(V(X, \lambda)) = \mathcal{D}(X)$). The union of green and red points forms the set $V_+(X, \lambda)$. Given a green or red point, the associated $I \in \mathbb{I}(X)$ corresponds to the black lines that go through the point. The basis $[x_4, x_6] \in \mathbb{B}(X')$ is the only one that is not obtained as an extension $\text{ex}(I)$ of some independent

$I \in \mathbb{I}(X)$; it corresponds to the hollow red point. Note that the order of the set X plays no role here. However, the order of B_0 is important: once we switch the order of x_5 and x_6 , the excluded basis becomes $[x_4, x_5]$. This changes the set $V_+(X, \lambda)$, as well as the space $\mathcal{D}_+(X)$.

As our results tell us, the dual geometry does not utilize the auxiliary basis B_0 :

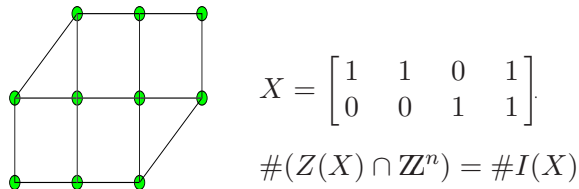


Figure 19. The lattice points of the zonotope $Z(X)$

There are 10 lattice points in the closed zonotope, but the association with the independent set is far less trivial. The proof of the relation

$$\mathcal{P}_+(X) = \Pi(\mathcal{Z}_+)$$

relied on the fact that the 10-set \mathcal{Z}_+ can be covered by a suitable number of translates of dual hyperplanes. There are three dual hyperplane here, and their multiplicity number $m(H)$ is either 2 (for H_1 , with $H_i := \text{span}\{x_i\}$), or 3 (for the other two choices). The figure below shows the covering of \mathcal{Z}_+ by translates of those hyperplanes.

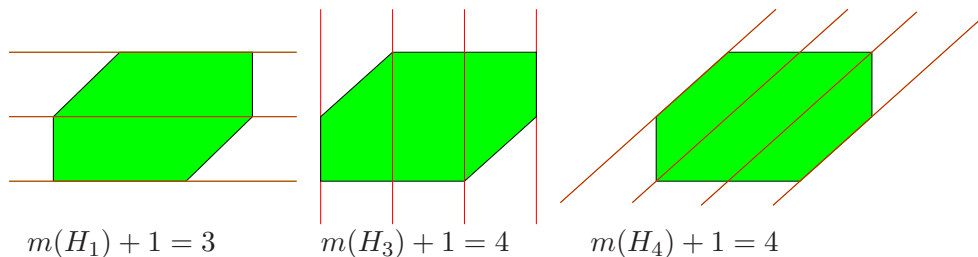


Figure 20. Covering of \mathcal{Z}_+ by dual hyperplanes

□

Sketch of the proof of the Lemma.

Step I. We append to X an auxiliary basis B_0 , and obtain $X' := X \cup B_0$ (we can choose B_0 any way we want since the definition of $I_+(X)$ does not make use of B_0 : by choosing B_0 to be equal (as a set) to an existing basis $B \in \mathbb{B}(X)$, we make the first part of the proof easier and the second part harder. By choosing B_0 to be in general position with respect to X , we make

the first part harder, but the second easier. Our actual proof uses the latter choice for B_0). It is possible to check that

$$I(X') \subset I_+(X).$$

Hence,

$$\ker I_+(X) \subset \ker I(X') = \mathcal{P}(X').$$

Step II. We order X' in a way that B_0 is placed after X , we let $(Q_B = q_{Y'_B})$ be the homogeneous basis for $\mathcal{P}(X')$ (per the given order). We define

$$\mathbb{B}' := \mathbb{B}(X') \setminus \mathbb{B}_+(X), \quad F := \text{span} \{ Q_B : B \in \mathbb{B}' \}.$$

We prove that $F \cap \ker I_+(X) = \{0\}$ which implies that the quotient map $\mathcal{P}(X') \rightarrow \mathcal{P}(X')/F$ is an injection on $\ker I_+(X)$, hence that

$$\dim \ker I_+(X) \leq \dim \mathcal{P}(X') - \dim F = \#\mathbb{B}_+(X) = \#\mathbb{I}(X).$$

Let

$$f := \sum_{B \in \mathbb{B}'} a(B) Q_B \in F.$$

Assume $f \in \ker I_+(X)$. We claim that $a(B) = 0, \quad \forall B \in \mathbb{B}'$. To this end, we grade the bases in \mathbb{B}' according to the location in B_0 of their maximal element (the maximal element must be in B_0 , since otherwise $B \in \mathbb{B}(X)$). “Location” is defined with respect to the given order of B_0 . Assume that there exists $B_1 \in \mathbb{B}'$ such that $a(B_1) \neq 0$. Assume further (without loss) that $a(B) = 0$ for every basis $B \in \mathbb{B}'$ with higher grade. We then choose $H := \text{span}(B \setminus \max(B))$, choose η to be normal to H , and choose a differential operator D_η^k , with

$$k := \#\{x \in X' \setminus H : x \notin \text{span}\{b \in B_1 : b \leq x\}\}.$$

One first shows that D_η^k annihilates $\ker I_+(X)$, hence annihilates f . One then shows that, selecting $\mathbb{B}'' \subset \mathbb{B}'$ to be the bases B that: (i) their grade is \leq that of B_1 , and (ii) $D_\eta^k Q_B \neq 0$, we have that

$$\{D_\eta^k Q_B : B \in \mathbb{B}''\}$$

is a set of linearly independent polynomials in $\mathcal{P}(X \cap H)$. This implies that $a(B) = 0, B \in \mathbb{B}''$, and leads to a contradiction, since $B_1 \in \mathbb{B}''$. \square

2 The external Hilbert function and graphs

We recall that the central Hilbert function of a graphical X is connected to the notions of parking functions and tree inversions. A similar connection exists in the external case. Moreover, while it is non-trivial to make this connection in the central case, it is quite straightforward to extend the connection to the external case. Let us draw this connection in the case of a complete graph G with $n + 1$ vertices $[0 : n]$.

Let's discuss first tree inversions. An independent set $I \in \mathbb{I}(X)$, X graphical, is a *forest* of the graph G , i.e., a subgraph without cycles. Given a forest I , and two vertices $0 \leq i < j \leq n$, we say that the edge (i, j) is an *inversion* in I if (i) i and j lie both in the same connected component I_0 of I , and (ii) the path in I that connects the minimal vertex of I_0 to i goes through j . An $(i, j) \in G$ that is not an inversion in I is *consistent* with I . The number of edges in $X = G$ that are consistent with I is denoted by $con(I)$. Then:

$$h_{G,+}(j) = \#\{I \in \mathbb{I}(G) : con(I) = j\}.$$

This observation follows easily from the corresponding one on spanning trees: one fixes a partition of the vertices of G into connected components and applies to each connected component the known connection between the central Hilbert function and inversions in spanning trees.

The connection with parking functions is done by suitably extending this notion. In the case of a complete graph the extension is done as follows (I believe the extension is known in the literature as “multi-parking function”; since I have not checked that yet, I'm using a different terminology): an *external parking function* K is a partition of the integers $[1 : n]$ into $1 \leq k \leq n$ subsets $(T_m)_{m=1}^k$, together with k parking functions that are defined on each T_m (i.e., with $i_m := \#T_m$, and $0 \leq i < i_m$, the parking function defined on T_m assumes values $\leq i$ more than i times). The value of the external parking function K is the sum of its entries together with $(1/2)((n + 1)^2 - \sum_{m=1}^k i_m^2)$. Then, for every j , $h_{G,+}(j)$ counts the number of external parking functions with value j .

3 Notes on this lecture

The count of lattice points in a unimodular zonotope follows from Stanley, 1973. The external theory as presented in the last two lectures is taken from Holtz, Ron, 2007.