Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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Our last lecture provides an overview (void of proofs) of the last layer of our 3-layer theory: the internal one.

1 Internal theory: the interior of a zonotope

We first recall the definition of internal bases. We choose an (arbitrary but fixed) ordering of X. Let $B \in \mathbb{B}(X)$. If, for each $b \in B$,

$$b \neq \max\{\, X\backslash H \,\}, \quad H := \operatorname{span}\{\, B\backslash b \,\} \in \mathcal{H}^*(X),$$

then B is called an *internal basis*. We denote the set of all internal bases by

$$\mathbb{B}_-(X).$$

For each $B \in \mathbb{B}(X)$, we define the *dual valuation* as follows:

$$\operatorname{val}^*(B) := \#\{\, b \in B \,:\, b \neq \max\{\, X\backslash \operatorname{span}(B\backslash b) \,\}\,\}.$$

Then,

$$\mathbb{B}_-(X) = \{\, B \in \mathbb{B}(X) \,:\, \operatorname{val}^*(B) = n \,\}.$$

Remarks. (i) In graph/matroid theory, if, for $b \in B$ and $H := \operatorname{span}(B\backslash b)$, we have that $b = \max X\backslash H$, then b is said to be *internally active* in B. The number $n - \operatorname{val}^*(B)$ is known as the *internal activity* of B. (ii) The valuation "val*" is truly dual to "val" in the sense that it coincides with the valuation "val" on the dual matroid of X. We make no use of this duality, primarily since the dual matroid of X is (usually) not vectorial. □

Fact:

$$\#\mathbb{B}_-(X) = \sum_{I\in\mathbb{I}(X)} (-1)^{n-\#I}.$$

Definition 1.1 *A subset $Y \subset X$ is said to be* **strongly of full rank** *if $Y\backslash y$ is full rank for any $y \in Y$. A subset $Y \subset X$ is* **very short** *if $X\backslash Y$ is strongly of full rank.*

We denote by

$$S_-(X)$$

the collection of very short subsets of X, and define

$$\mathcal{P}_-(X) := \operatorname{span}\{\, q_Y \;:\; Y \in S_-(X)\,\},$$

and

$$I_-(X) := \operatorname{Ideal}\left\{ q_{\eta_H}^{m(H)-1} \;:\; \eta_H \perp H, \quad H \in \mathcal{H}^*(X) \right\}.$$

Note that $\mathcal{P}_-(X)$ and $\#\mathbb{B}_-(X)$ are independent of the order we choose for X. Needless to say, the set $\mathbb{B}_-(X)$ itself depends on that order.

The **internal Hilbert function** $h_{X,-}$ records the homogeneous dimensions of $\mathcal{P}_-(X)$:

$$h_-(j) := h_{X,-}(j) := \dim(\mathcal{P}_-(X) \cap \Pi_j^0), \quad j \in \mathbb{N}.$$

Theorem 1.2

(1) $\mathcal{P}_-(X) = \ker I_-(X)$.

(2) $\dim \mathcal{P}_-(X) = \#\mathbb{B}_-(X)$.

(3) For $j = 0, 1, 2, \ldots,$

$$h_-(j) = \#\{B \in \mathbb{B}_-(X) :\ \operatorname{val}(B) = j\}.$$

□

Note that we do *not* claim that the polynomials $Q_B := q_{Y_B}$, $B \in \mathbb{B}_-(X)$, that are (being a part of the homogeneous basis form $\mathcal{P}(X)$) necessarily linearly independent, form a basis for $\mathcal{P}_-(X)$. Such a claim is true only $n = 2$. For $n = 3$, one can construct a counterexample already with $\#X = 5$:

Example 1.3 *Let*

$$X = [x_1, \ldots, x_5] := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then, a 4-set $Y \subset X$ is strongly of full rank iff it contains x_1 and x_3. At the same time,

$$\mathbb{B}_-(X) = \{[x_1, x_2, x_3], [x_1, x_3, x_4], [x_1, x_2, x_4]\} =: (B_1, B_2, B_3).$$

The above theorem asserts, then, that $\dim \mathcal{P}_-(X) = 3$. *Indeed, one verifies directly that*

$$\mathcal{P}_-(X) = \operatorname{span}\{1, q_{x_2}, q_{x_4}\}.$$

The polynomials $Q_{B_1} = 1$, $Q_{B_2} = q_{x_2}$ and $Q_{B_3} = q_{x_3}$ do not form a basis for $\mathcal{P}_-(X)$, since $Y_{B_3} = \{x_3\}$ is not very short (its complement, $\{x_1, x_2, x_4, x_5\}$, is not strongly of full rank). However, the above three polynomials do capture correctly the homogeneous dimensions of $\mathcal{P}_-(X)$, hence can be used for computing h_-, as asserted in the theorem. □

The connection here to zonotopes is as follows:

Theorem 1.4 *Let $\mathcal{Z}_- := \mathcal{Z}_-(X)$ be the integer points in the interior of the zonotope $Z(X)$. Then*

$$\Pi(\mathcal{Z}_-) = \mathcal{P}_-(X),$$

provided that X is unimodular.

The *proof* of this last assertion is very similar to the proofs of the analogous results in the central and external theories. First, one verifies that the cardinality of $\mathcal{Z}_-$ equals $\dim \mathcal{P}_-(X)$ (which already requires the unimodularity of X), and then one uses the identity $\mathcal{P}_-(X) = \ker I_-(X)$: this identity reduces the proof to showing that, given any $H \in \mathcal{H}^*(X)$, the set $\mathcal{Z}_-$ lies in the union of $m(H) - 1$ translates of H.

2 Internal theory - the bounded regions of a hyperplane arrangement

Once again we order X, and define the set of *barely long subsets of* X:

$$L_-(X) := \{\, Y \subset X \;:\; Y \cap B \neq \emptyset, \quad B \in \mathbb{B}_-(X) \,\}.$$

The corresponding ideal is

$$J_-(X) := \operatorname{Ideal}\{\, q_Y \;:\; Y \in L_-(X) \,\},$$

and the notation for its kernel is

$$\mathcal{D}_-(X) := \ker J_-(X).$$

It is clear that $J_-(X) \supset J(X)$ hence that $\mathcal{D}_-(X) \subset \mathcal{D}(X)$. As is the case with external theory, one easily finds that $\mathcal{D}_-(X)$ depends on the ordering of X. (To recall, $\mathcal{D}(X)$ does not depend on any ordering.)

Theorem 2.1

(1) The map $p \mapsto \langle p, \cdot\rangle$ *is a bijection between* $\mathcal{D}_-(X)$ *and* $\mathcal{P}_-(X)'$*, in particular:*

(2) $\dim \mathcal{D}_-(X) = \#\mathbb{B}_-(X)$,

(3) $\dim\left(\mathcal{D}_-(X) \cap \Pi_j^0\right) = h_-(j)$, $j \in \mathbb{N}$, *and*

(4) $\Pi = \mathcal{P}_-(X) \oplus J_-(X)$. □

Example 2.2 Let

$$X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} = [x_1, x_2, x_3, x_4].$$

Then

$$\mathbb{B}_-(X) = \{\{x_1, x_2\},\ \{x_1, x_3\}\},$$
$$L_-(X) = \{\{x_1\},\ \{x_2, x_3\}, \cdots\},$$
$$J_-(X) = \text{Ideal}\left\{q_{x_1}, q_{[x_2,x_3]}\right\},$$
$$\ker J_-(X) = \mathcal{D}_-(X) = \text{span}\{1, q_{x_5}\}, \quad x_5 := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$S_-(X) = \{\{x_2\},\ \emptyset\},$$
$$I_-(X) = \text{Ideal}\left\{q_{x_3},\ q_{x_2}^2,\ q_{x_5}^2\right\},$$
$$\ker I_-(X) = \text{span}\{1,\ q_{x_2}\} = \mathcal{P}_-(X). \qquad \square$$

Finally, the space $\mathcal{D}_-(X)$ is connected to the *internal vertices* of the hyperplane arrangement: given a generic $\lambda \in \mathbb{C}^X$, we let

$$V_-(X,\lambda) := \{v_B \in V(X,\lambda) : B \in \mathbb{B}_-(X)\}.$$

Then:

Theorem 2.3

$$\Pi\left(V_-(X,\lambda)\right) = \mathcal{D}_-(X).$$

□

3 Notes on this lecture

The lecture is taken from Holtz, Ron, 2007. As far as the proofs are concerned, there are two that are quite non-trivial: the proof that $J_-(X)+\mathcal{P}_-(X) = \Pi$, and the proof that $\dim\ker I_-(X) \leq \#\mathbb{B}_-(X)$.