Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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Lecture 2: Deriving polynomial spaces from the geometry (2 February 2007)

1 Notations.

Let $\sigma \subset \mathbb{C}^n$ be a finite subset (not a multiset). We are looking for a polynomial subspace

$$\Pi(\sigma) \subset \mathbb{C}[t_1, \cdots t_n] =: \Pi$$

such that the restriction map $\Pi(\sigma) \to \mathbb{C}^\sigma \,:\, p \mapsto p|_\sigma$ is an isomorphism. In particular, we must have $\dim \Pi(\sigma) = \#\sigma$.

To this end, we look at the point evaluation functional

$$\delta_\alpha : \Pi \to \mathbb{C} \,:\, p \mapsto p(\alpha),$$

and define

$$\Lambda := \operatorname{span}\{\, \delta_\alpha \,:\, \alpha \in \sigma \,\}.$$

Then Λ is a subspace of the dual space Π' of Π.

2 The polynomial space $\Pi(\sigma)$.

Our goal can now be restated as follows: For given σ, we want to find $F \subset \Pi$ such that

$$\Lambda|_F \cong F',$$

where F' is the dual space of F. We approach this problem by associating each $p \in \Pi$ with a differential operator:

$$p(D) := p\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_n}\right).$$

Thus, with $q_x(t) := x \cdot t, \quad x \in \mathbb{R}^n \backslash \{ 0 \}$, we get that $q_x(D) = D_x$, the directional derivative in the x-direction.
Next, for an entire function f and a polynomial p, we define a pairing:

$$\langle p, f \rangle := p(D) f(0).$$

The representation of δ_α in this pairing is done by the exponential $e_\alpha : \mathbb{R}^n \to \mathbb{R} : t \mapsto e^{\alpha \cdot t}$; i.e.,

$$\langle p, e_\alpha \rangle = p(\alpha) = \delta_\alpha p.$$

Thus the abstract space Λ is represented here concretely by the exponential space

$$\mathrm{Exp}(\sigma) := \mathrm{span} \{ e_\alpha : \alpha \in \sigma \} .$$

This allows us to look at the power expansion of the exponentials (which is not a big deal: the coefficients in the power expansion are essentially the action of δ_α on the monomial basis for Π. The only "real achievement" here that we get for free the correct normalization of the monomials that works for us). For each $f \in \mathrm{Exp}(\sigma)$, we write the power series expansion of f in the form

$$f = f_0 + f_1 + f_2 + \cdots ,$$

where f_j is a homogeneous polynomial of degree j.
We then define a non-linear action $f \mapsto f_\downarrow$ by

$$f_\downarrow := f_j, \quad f_j \neq 0, \ f_i = 0, \ i < j.$$

I.e., $f_\downarrow$ is the first non-zero term in the above expansion of f ($0_\downarrow := 0$). Finally, we define

$$\Pi(\sigma) := \mathrm{span} \{ f_\downarrow : f \in \mathrm{Exp}(\sigma) \} .$$

Example 2.1

$$(e_\alpha)_\downarrow = 1,$$
$$(e_\alpha - e_{\alpha'})_\downarrow = q_{\alpha - \alpha'} .$$

□

Question: What is the dimension of $\Pi(\sigma)$?

For an entire function f, let $T_j f$ be the j-th degree Taylor expansion of f at 0 (T_j is obviously linear):

$$T_j f := f_0 + f_1 + \cdots + f_j.$$

Note that $\deg(f_\downarrow) = j$ iff $f \in \ker T_{j-1} \backslash \ker T_j$. Thus, $\dim \operatorname{span}\{f_\downarrow : f \in \operatorname{Exp}(\sigma),\ \deg(f_\downarrow) = j\}$ is, with T'_j the restriction of T_j to $\ker T_{j-1}$,

$$\operatorname{rank} T'_j = \dim \ker T_{j-1} - \dim \ker T_j.$$

Summing from $j = 0$ to ∞ (where $T_{-1} := 0$), we easily obtain that

$$\sum_{j=0}^{\infty} \dim(\ker T_{j-1}) - \dim(\ker T_j) = \dim(\ker T_{-1}) = \dim \operatorname{Exp}(\sigma) = \#\sigma.$$

Here we used the fact that every finite set of exponentials is linearly independent.

Note that, for any analytic function $f \neq 0$,

$$\langle f_\downarrow, f \rangle \neq 0.$$

This means that there exists no $f \in \operatorname{Exp}(\sigma) \backslash \{0\}$ that satisfies

$$\langle p, f \rangle = 0, \quad \forall p \in \Pi(\sigma);$$

(choose $p := f_\downarrow \in \Pi(\sigma)$). Thus,

$$\dim \Pi(\sigma) = \dim \operatorname{Exp}(\sigma),$$

and no non-zero linear functional $f \in \operatorname{Exp}(\sigma)$ vanishes on $\Pi(\sigma)$. Thus we proved:

Theorem 2.2 *The map* $\operatorname{Exp}(\sigma) \to \Pi(\sigma)'$ *given by the above pairing is an isomorphism, hence* $\Pi(\sigma)$ *solves our original problem.*

Note that the pairing $\langle \cdot, \cdot \rangle$ allows us to identify any entire function f as a functional in Π'. Also, the action $f \mapsto f_\downarrow$ is defined for every entire function, hence the definition

$$F_\downarrow := \operatorname{span}\{\, f_\downarrow \ :\ f \in F \,\}$$

makes sense for any space of entire functions. This explains the meaning of the left-hand side space in our next theorem (albeit, a closer scrutiny reveals that the left-hand-side space is still an exponential space).

Now, given any non-zero $p \in \Pi$, let $p_\uparrow$ to be the highest degree homogeneous polynomial in p, i.e., $p_\uparrow$ is homogeneous, and $\deg(p - p_\uparrow) < \deg p$. Then we have the following theorem:

Theorem 2.3 *Let* $p \in \Pi$ *and let* $\sigma \subset \mathbb{C}^n$ *be finite. Then*

$$(p(D)\,(\operatorname{Exp}(\sigma)))_\downarrow \supset p_\uparrow(D)(\Pi(\sigma)).$$

$\square$

The theorem can be used as follows: suppose that we hold in hand a homogeneous polynomial q, and we would like to understand the action of $q(D)$ on $\Pi(\sigma)$. The way to go is to try to find a non-homogeneous polynomial p such that (i) $p_{\uparrow} = q$, and (ii) p vanishes at as many points as possible of σ. Indeed, one easily verifies that

$$p(D)\mathrm{Exp}(\sigma) = \mathrm{Exp}(\sigma \backslash Z_p),$$

with Z_p the zero-set of p. Thus, the above theorem, in particular, tells us that

$$\dim(q(D)\Pi(\sigma)) \le \#(\sigma \backslash Z_p),$$

with p any *lower order perturbation* of q, as above.

As a special case, we get the following important corollary:

Theorem 2.4 *If p vanishes on σ then $p_{\uparrow}(D)$ annihilates $\Pi(\sigma)$.* □

Setup: The theorem gives us a handy tool for estimating below dimensions of spaces of the form

$$D = \{f \in \Pi : q(D)f = 0, \ \forall q \in J\},$$

with J some finite set of homogeneous polynomials. (We refer to spaces as the above D as (joint) kernels of differential operators). The idea is as follows, for each $q \in J$, we define (very carefully!!) a polynomial p_q such that $p_{q\uparrow} = p$. Then we find the common zero set σ of the polynomials $(p_q : q \in J)$. Since each p_q vanishes on σ, then, by the above theorem, each $q(D)$ annihilates $\Pi(\sigma)$, hence

$$\Pi(\sigma) \subset D.$$

Since $\dim \Pi(\sigma) = \#\sigma$, we obtain in this way the lower bound

$$\dim D \ge \#\sigma.$$

We conclude with a concrete example (actually, two).

Example 2.5

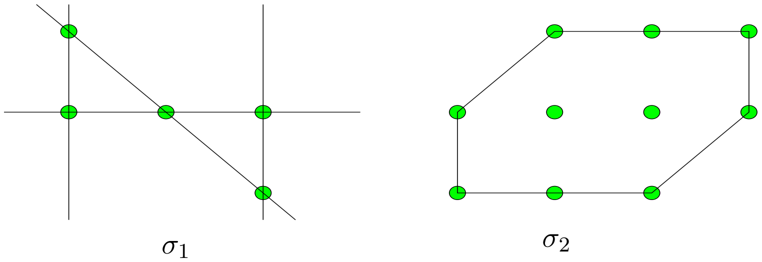
σ_1 σ_2

Figure 8: Computing $\Pi(\sigma)$ in two concrete examples

For σ_1: direct computation yields that

$$\mathrm{Exp}(\sigma_1) = \mathrm{span}\left\{ 1, e^{t_1}, e^{t_2}, e^{2t_1-t_2}, e^{2t_1} \right\}, \quad \Pi(\sigma_1) = \mathrm{span}\left\{ 1,\ t_1,\ t_2,\ t_1^2,\ t_2^2 - 2t_1t_2 \right\}.$$

Let's employ now the tools developed in this lecture; we use the notations as in Example 1 of Lecture 1. One observes that each of the polynomials

$$p_{x_3}p_{x_4}, \quad p_{x_1}p_{x_2}p_{x_3}, \quad p_{x_1}p_{x_2}p_{x-4},$$

vanishes on σ_1. Therefore, each the three differential operators

$$D_{x_3}D_{x_4}, \quad D_{x_1}D_{x_2}D_{x_3}, \quad D_{x_1}D_{x_2}D_{x-4},$$

must annihilate $\Pi(\sigma_1)$. In particular the joint kernel of these operators is of dimension ≥ 5. In fact, $\Pi(\sigma_1)$ is *exactly* that joint kernel, but the tools from the current lecture fall short of showing this latter fact.

For σ_2: direct computation yields that

$$\mathrm{Exp}(\sigma_2) = \mathrm{span}\left\{ 1, e^{t_1}, e^{t_2}, e^{2t_1}, e^{t_1+t_2}, e^{2t_1+t_2}, e^{3t_1+t_2}, e^{t_1+2t_2}, e^{2t_1+2t_2}, e^{3t_1+2t_2} \right\},$$

and

$$\Pi(\sigma_2) = \mathrm{span}\left\{ 1,\ t_1,\ t_2,\ t_1^2,\ t_1t_2,\ t_2^2,\ t_1^2t_2,\ t_1t_2^2,\ t_1^3,\ t_1^2t_2(t_1+t_2) \right\}.$$

The polynomial

$$q_{x_2}(q_{x_2}-1)(q_{x_2}-2)$$

vanishes on σ_2. Therefore, the differential operator $D_{x_2}^3$ annihilates $\Pi(\sigma_2)$. Using similar reasonings, the operators

$$D_{x_1}^4 \text{ and } D_{x_5}^4, \quad x_5 := [1 -1]',$$

annihilate $\Pi(\sigma_1)$ as well, hence the joint kernel of these three operators is of dimension ≥ 10. Again, this dimension is exactly 10 (hence $\Pi(\sigma_2)$ is exactly the joint kernel).

The reader might wish to revisit this example after reading further into these lectures. In terms of future notations, and with X as in Example 1, the space $\Pi(\sigma_1)$ is $\mathcal{D}(X)$, while $\Pi(\sigma_2)$ is $\mathcal{P}_+(X)$. The dimension 5 of the former equals $\#\mathbb{B}(X)$, while the dimension 10 of the latter equals $\#\mathbb{I}(X)$.

3 Hilbert functions

Hilbert functions or series are usually associated with underlying ideals. In our setup, there is an equivalent approach that allows us to define those functions without even defining the notion of an ideal. This recourse goes as follows. First, let us denote by

$$\Pi_j^0 \subset \Pi$$

the subspace of all *homogeneous* polynomials of degree j. Given a finite $\sigma \subset \mathbb{C}^n$, we define

$$h : \mathbb{N} \to \mathbb{N} : j \mapsto \dim(\Pi(\sigma) \cap \Pi_j^0).$$

In the case of σ_1 above, the Hilbert function is thus

$$h = (1, 2, 2),$$

while the Hilbert function of σ_2 is

$$h = (1, 2, 3, 3, 1).$$

Note that σ_2 comprises the lattice points in a unimodular zonotope. We will see (in Lecture 9) that the Hilbert function of the lattice points in every unimodular zonotope ends with 1 (more precisely, 1 is the last non-zero entry).

Though it is not entirely obvious, one should note that the Hilbert function associated with the pointset σ is nothing but the Hilbert function of the ideal of polynomials that vanish on σ.

4 Notes on this lecture

This lecture, as a whole, is taken from the following two papers:

C. de Boor, A. Ron, On multivariate polynomial interpolation, Constructive Approximation **6**(1990), 287–302.

C. de Boor, A. Ron, On ideals of finite codimension and applications to box splines theory, Journal of Mathematical Analysis and its Applications **158** (1991), 168–193.

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