Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

Lectures given by Amos Ron
and recorded by Yeon Hyang Kim

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1 Our basic tool.

We stated last time the following result:

Corollary 1.1 *Let* $\sigma \subset \mathbb{R}^n$ *be a finite subset. Let* $p \in \Pi$. *If* $p|_\sigma = 0$, *then* $p_\uparrow(D)(\Pi(\sigma)) = 0$.

Here is the (strikingly simple) proof for this result:
Proof: Set $p =: p_\uparrow + p'$; then $\deg p' < \deg p$. For $f \in \mathrm{Exp}(\sigma)$, set $f =: f_\downarrow + f'$. The assumption $p = 0$ on σ is equivalent to

$$p(D)(\mathrm{Exp}(\sigma)) = 0.$$

Consequently,

$$p(D)f = 0 = p_\uparrow(D)f_\downarrow + \text{ h.o.t.},$$

with "h.o.t" meaning for "higher order terms". Therefore,

$$p_\uparrow(D)f_\downarrow = 0.$$

Thus, $p_\uparrow(D)$ annihilates a spanning set of $\Pi(\sigma)$, hence annihilates all functions in that space. □

2 The space $D_{\mathbb{B}'}(X)$.

We now depart the general discussion of spaces of the form $\Pi(\sigma)$, and turn our attention back to the multiset X, and its associated hyperplane arrangement $\mathcal{H}(X, \lambda)$. Recall that, for a generic λ, there is a natural bijection $B \mapsto v_B$ between the vertex set $V(X, \lambda)$ of the arrangement, and the basis set $\mathbb{B}(X)$. In particular,

$$\#V(X) = \#\mathbb{B}(X).$$

For a multi-subset $Y \subset X$, we define

$$p_Y := \prod_{y \in Y} p_y,$$

$$q_Y := \prod_{y \in Y} q_y,$$

where $q_y : \mathbb{R}^n \to \mathbb{R} : t \mapsto y \cdot t$, and $p_y = q_y - \lambda_y$.

Now, let us choose a subset $\mathbb{B}'$ of $\mathbb{B}(X)$. The subset is arbitrary, but fixed. Let $V' \subset V(X, \lambda)$ be the set of vertices of the hyperplane arrangement associated with $\mathbb{B}'$. We associate $\mathbb{B}'$ with the following space of polynomials:

$$\mathcal{D}_{\mathbb{B}'}(X) := \left\{ f \in \Pi \, : \, q_Y(D)f = 0, \quad \forall Y \subset X \text{ such that } Y \cap B \neq \emptyset, \ \forall B \in \mathbb{B}' \right\}.$$

Then we have the following remarkable theorem (which is “remarkable” since no assumption is made on $\mathbb{B}'$):

Theorem 2.1 *Let $X \subset \mathbb{R}^n \backslash \{0\}$ be a finite multiset of full rank n. Then, for any $\mathbb{B}'$ of $\mathbb{B}(X)$,*

$$\dim \mathcal{D}_{\mathbb{B}'}(X) \geq \#\mathbb{B}'.$$

Proof: Note that for $Y \subset X$ and $B \in \mathbb{B}(X)$, $p_Y(v_B) = 0$ iff $Y \cap B \neq \emptyset$ (since p_Y vanishes on the union of hyperplanes $H_{x,\lambda}$, $x \in Y$, while v_B is the intersection of the hyperplanes $H_{b,\lambda}$, $b \in B$). Now, set $V' := \{v_B : \ B \in \mathbb{B}'\}$, and let Y be a multi-subset of X such that

$$Y \cap B \neq \emptyset, \ \forall B \in \mathbb{B}'.$$

Since p_Y vanishes on V', then, by Theorem 1.1, we have

$$((p_Y)_\uparrow)(D)(\Pi(V')) = 0.$$

Since $(p_Y)_\uparrow = q_Y$, we conclude that

$$\Pi(V') \subset \mathcal{D}_{\mathbb{B}'}(X).$$

But,

$$\dim \Pi(V') = \#V' = \#\mathbb{B}'.$$

□

The proof above supports the following corollary:

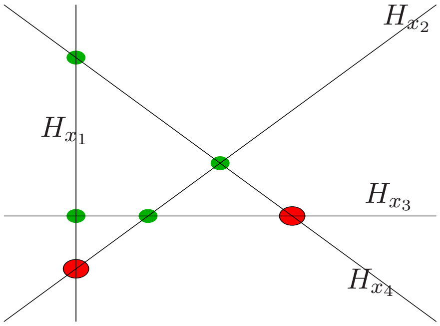
Corollary 2.2 *With $\mathbb{B}'$, $\mathcal{D}_{\mathbb{B}'}(X)$, and V' as above, we have that*

$$\mathcal{D}_{\mathbb{B}'}(X) = \Pi(V')$$

if and only if $\dim \mathcal{D}_{\mathbb{B}'}(X) \leq \#\mathbb{B}'$.

Example 2.3 Let

$$X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} =: [x_1, x_2, x_3, x_4], \quad \mathbb{B}' = \{\, [x_1\ x_2],\ [x_3\ x_4]\,\},$$



$$\begin{aligned} p_{x_1}(t) &:= t_1 \\ p_{x_2}(t) &:= t_1 - t_2 - 1 \\ p_{x_3}(t) &:= t_2 \\ p_{x_4}(t) &:= t_1 + t_2 - 3 \end{aligned}$$

Figure 9: Hyperplanes of X

The green+red dots comprise the vertex set $V(X, \lambda)$, while the red dots correspond to V'. Here, for $Y \subset X$ to intersect both bases in $\mathbb{B}'$, we clearly must have $\#Y \geq 2$. Thus, for any such Y, $q_Y(D)$ annihilates Π_1 (the space of linear polynomials in two variables) hence

$$\dim D_{\mathbb{B}'}(X) \geq 3 > 2 = \#\mathbb{B}'.$$

$\square$

3 $\mathcal{D}(X)$ and $\mathcal{P}(X)$: the central spaces of the central theory

We have seen the geometric duality between the hyperplane arrangement and the zonotope that are associated with the multiset X. We will now develop an algebraic counterpart of that duality. The algebraic structures will come in three pairs of polynomials spaces. Each pair will be shown to be dual space of its pairmate via our pairing $\langle \cdot, \cdot \rangle$.

The first pair will be referred to as *the central pair* of X. The space $\mathcal{P}(X)$ below is the central space of the zonotope $Z(X)$, while the space $\mathcal{D}(X)$ is the central space of the hyperplane arrangement.

In order to define those two spaces, we decompose the power set 2^X into the following two disjoint subsets: the set of long subsets ($L(X)$) and the set of short subsets ($S(X)$):

$$L(X) := \{\, Y \subset X \, : \, Y \cap B \neq \emptyset, \quad \forall B \in \mathbb{B}(X)\,\} = \{\, Y \subset X \, : \, \operatorname{rank}(X \backslash Y) < n\,\},$$

and

$$S(X) := \{\, Y \subset X \, : \, \operatorname{rank}(X \backslash Y) = n\,\} = 2^X \backslash L(X).$$

We then define

$$\mathcal{D}(X) := \{\, p \in \Pi \,:\, q_Y(D)p = 0, \quad \forall Y \in L(X) \,\},$$

and

$$\mathcal{P}(X) := \operatorname{span}\{\, q_Y \,:\, Y \in S(X) \,\}.$$

Thus, $\mathcal{D}(X)$ is "the kernel of the long subsets" while $\mathcal{P}(X)$ is "the span of the short subsets". This will be the rule all the way through: $\mathcal{D}$-type spaces are joint kernels of differential operators that are defined by long subsets, while $\mathcal{P}$-type spaces are the span of polynomials that are defined by short subsets. The precise definitions of "long" and "short" depend on the context. In the context of the central theory, the definitions are as above.

Theorem 3.1

(1) $\dim \mathcal{D}(X) = \dim \mathcal{P}(X) = \#\mathbb{B}(X)$,

(2) The map

$$\mathcal{D}(X) \to \mathcal{P}(X)' \,:\, p \mapsto \langle p, \cdot \rangle$$

is an isomorphism.

(3) $\Pi(V(X,\lambda)) = \mathcal{D}(X)$*, for every generic* λ*.*

Note that, by the results of Lecture 2, (3) above follows from the fact that $\dim \mathcal{D}(X) = \#\mathbb{B}(X)$. Moreover, since we already know that $\dim \mathcal{D}(X) \geq \#\mathbb{B}(X)$ (see Lecture 2), it suffices to prove the following: (i) the map in (2) is injective (and thus $\dim \mathcal{D}(X) \leq \dim \mathcal{P}(X)$, and (ii) $\dim \mathcal{P}(X) \leq \#B(X)$.

4 Combinatorics: the central Hilbert function of X

Let Π_j^0 be the space of homogeneous polynomials of degree j (in n variables). The asserted isomorphism (part (2)) in the theorem above implies (since both $\mathcal{P}(X)$ and $\mathcal{D}(X)$ are *homogeneous*, i.e., are spanned by homogeneous polynomials), that, for every j

$$\dim(\Pi_j^0 \cap \mathcal{D}(X)) = \dim(\Pi_j^0 \cap \mathcal{P}(X))$$

We refer to the homogeneous dimensions of the space $\mathcal{P}(X)$ as the *central Hilbert function of* X, i.e.,

$$h_X : \mathbb{Z}_+ \to \mathbb{Z}_+ \,:\, j \mapsto \dim(\Pi_j^0 \cap \mathcal{P}(X)).$$

Note that

$$\sum_j h_X(j) = \#\mathbb{B}(X).$$

In view of the theorem above, h_X is the Hilbert function of the ideal of polynomials that vanish on the vertex set $V(X, \lambda)$ of the hyperplane arrangement. In particular, this latter Hilbert function does not depend on the choice of λ (for as long as λ is generic).

We will show later that h_X can be computed directly by studying the dependence/independence relations among the vectors in X, and is connected with the notion of *external activity* in graph/matroid theory.

Example 4.1 *Let*

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}.$$

Then $\#\mathbb{B}(X) = 16$*, and with some effort the Hilbert function can be shown to be*

$$h_X = (1, 3, 6, 6).$$

$\square$

5 Notes on Lectures 2 and 3

Lecture 2, as a whole, is taken from the following two papers:

C. de Boor, A. Ron, On multivariate polynomial interpolation, Constructive Approximation **6***(1990), 287–302.*

C. de Boor, A. Ron, On ideals of finite codimension and applications to box splines theory, Journal of Mathematical Analysis and its Applications **158** *(1991), 168–193.*

Theorem 2.1 in the present lecture, as well as the given proof are taken from the latter paper.

Obviously, the theorem implies as a special case that

$$\dim \mathcal{D}(X) \geq \#\mathbb{B}(X).$$

This specific inequality was first proved by Dahmen and Micchelli (Studia Math., 1985) by induction on $\#X$ and on n. A non-inductive analytic-type argument appears in Ben Artzi and Ron (TAMS, 1988).

The assertion in Theorem 3.1 that $\dim \mathcal{D}(X) = \#B(X)$ is due to Dahmen and Micchelli (Studia Math., 1985). They subsequently provided (in Adv. Math., 1989) a very elegant proof for the inequality $\dim \mathcal{D}(X) \leq \#\mathbb{B}(X)$, that uses the matroidal structure of X.

The space $\mathcal{P}(X)$ was introduced independently by Hakopian and Sahakian (1988), and by Dyn and Ron (TAMS, 1990). The central Hilbert function is introduced in the last section of the latter paper.