Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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1 The duality between $\mathcal{D}(X)$ and $\mathcal{P}(X)$

Theorem 1.1

1. $\dim \mathcal{D}(X) = \dim \mathcal{P}(X) = \#\mathbb{B}(X)$,

2. *The map*

$$\mathcal{D}(X) \to \mathcal{P}(X)' \,:\, p \mapsto \langle p, \cdot \rangle$$

is an isomorphism.

Proof. For the proof, we will show that

$$\#\mathbb{B}(X) \le \dim \mathcal{D}(X) \le \dim \mathcal{P}(X) \le \#\mathbb{B}(X).$$

The left-most inequality was proved in Lecture 3. We will next prove that the map in the second part of the theorem is injective. This will also imply that $\dim \mathcal{D}(X) \le \dim \mathcal{P}(X)$.

Suppose the map in the second part of the theorem is not injective. Then there exists a polynomial $p \in \mathcal{D}(X)\backslash 0$ such that for any $Y \in S(X)$,

$$q_Y(D)p(0) = 0. \tag{1}$$

Without loss, we may assume that p is of minimal degree (among all non-zero polynomials that satisfy the above). Then we claim that for any $a \in \mathbb{R}^n\backslash 0$, $D_a p$ is another polynomial with the same property. To this end, we need to show that

$$q_Y(D)(D_a p)(0) = 0, \quad \forall Y \in S(X).$$

Note that $\mathcal{D}(X)$, as any joint kernel of differential operators with constant coefficients, is D-invariant, i.e., closed under differentiation (= translation invariant). Fix $Y \in S(X)$. Let $B \subset X\backslash Y$ be a basis. Then we can write

$$a = \sum_{b \in B} c_b b, \quad (c_b)_b \subset \mathbb{R}.$$

Now, since $B \cap Y = \emptyset$,

$$q_Y(D)(D_a p)(0) = \sum_{b \in B} c_b q_{Y \cup b}(D)p(0).$$

If $Y \cup b \in L(X)$, then $q_{Y \cup b}(D)p = 0$ since $p \in \mathcal{D}(X)$. Otherwise, $Y \cup b \in S(X)$, and hence, by (1), we also have $q_{Y \cup b}(D)p(0) = 0$. Hence, $D_a p$ is another polynomial with the same property as in (1), and the degree minimality assumption on p implies that $D_a p = 0$. Since $a \in \mathbb{R}^n\backslash 0$ is arbitrary, p is constant. But this is a contradiction, since $p \neq 0$ by assumption, and a non-zero constant p cannot satisfy (1) (since $\mathcal{D}(X)$ contains the constants). This establishes the requisite injectivity.

It remains, thus, to prove that

$$\dim \mathcal{P}(X) \leq \#\mathbb{B}(X).$$

Recall that we associate each $x \in X$ with a non homogeneous polynomial p_x:

$$p_x : \mathbb{R}^n \to \mathbb{R} : t \mapsto x \cdot t - \lambda_x,$$

where (λ_x) is chosen so that the intersection of any collection of $n+1$ hyperplanes is empty. First, we want to show that for any $Y \in S(X)$

$$p_Y := \Pi_{y \in Y} p_y \in \operatorname{span}\left\{ p_{X\backslash B} \ : \ B \in \mathbb{B}(X) \right\}. \tag{2}$$

We will prove the above by induction on $\#(X\backslash Y)$. If $\#Y = \#X - n$, then (2) is true. Assume thus that $\#Y < \#X - n$. Then, since $Y \in S(X)$ and $\#Y < \#X - n$, we can find a basis $B \subset X\backslash Y$, as well as an additional element $\xi \in (X\backslash(Y \cup B))$. Since $Z := B \cup \{ \xi \}$ is a dependent set, there exist coefficients $(c_z)_{z \in Z}$ such that

$$\sum_{z \in Z} c_z z = 0. \tag{3}$$

Then

$$\sum_{z \in Z} c_z p_z = -\sum_{z \in Z} c_z \lambda_z \neq 0.$$

(Indeed, let A be the $n+1 \times n+1$ matrix whose columns are indexed by Z, and whose zth column, $z \in Z$, is the vector (z, λ_z). Our generic choice of $(\lambda_x)_{x \in X}$ implies that the row-rank of this matrix is $n+1$, while the condition we need above is that its column rank is $n+1$.) W.L.o.G. we may assume that $\sum_{z \in Z} c_z \lambda_z = -1$. Thus,

$$p_Y \cdot 1 = p_Y \sum_{z \in Z} c_z p_z = \sum_{z \in Z} c_z p_{Y \cup z}.$$

Now, it is easy to see that, if $c_z \neq 0$, then $Y \cup \{ z \} \in S(X)$. Thus, by the induction assumption,

$$c_\xi p_{Y\cup\xi} + \sum_{b\in B} c_b p_{Y\cup b} \in \operatorname{span} \left\{ p_{X\backslash B} \ : \ B \in \mathbb{B}(X) \right\}.$$

This completes the induction, and, consequently,

$$\operatorname{span}\{p_Y : Y \in S(X)\} \subset \operatorname{span} \left\{ p_{X\backslash B} \ : \ B \in \mathbb{B}(X) \right\}.$$

We will show now that

$$\mathcal{P}(X) \subset \operatorname{span}\{p_Y : Y \in S(X)\},$$

and do that by using induction on $\#Y$ to show that

$$q_Y \in \operatorname{span}\{p_Z : Z \in S(X)\}, \quad Y \in \mathcal{S}(X).$$

The claim is trivial for $Y = \emptyset$, since then $p_Y = q_Y = 1$. The induction step is straightforward, since it is easy to see that

$$p_Y - q_Y \in \operatorname{span}\{q_Z : \ Z \subset X, \ \#Z < \#Y\}.$$

(Indeed,

$$p_Y(t) = \Pi_{y\in Y}(y \cdot t - \lambda_y) = q_Y(t) + \sum_{Z \subsetneq Y} c_Z q_Z(t).)$$

In summary, we have showed that

$$\mathcal{P}(X) \subset \operatorname{span} \left\{ p_{X\backslash B} \ : \ B \in \mathbb{B}(X) \right\},$$

hence, in particular,

$$\dim \mathcal{P}(X) \leq \#\mathbb{B}(X).$$

$\square$

Corollary 1.2 $\left\{ p_{X\backslash B} \ : \ B \in \mathbb{B}(X) \right\}$ *forms an (inhomogeneous) basis for* $\mathcal{P}(X)$.

$\square$

Discussion. We refer to the above basis as a **cobasis basis** for $\mathcal{P}(X)$ (in recognition of the fact that the set $\{ X\backslash B : \ B \in \mathbb{B}(X) \}$ is the basis set of the dual matroid).
Recall that the construction works only for a generic choice of

$$\lambda := (\lambda_x)_{x\in X}.$$

The cobasis basis is Lagrangian in the sense that, with $B \mapsto v_B$ the bijection between $\mathbb{B}(X)$ and the vertex set $V(X)$, the polynomial $p_{X\backslash B}$ vanishes on the entire $V(X)$ set with the exception of the vertex v_B. $\square$

In our next corollary we retain the assumption that the hyperplane arrangement $H(X)$ (which is induced by the zero sets H_x of the polynomials p_x, $x \in X$) is generic: $\#B(X) = \#V(X)$.

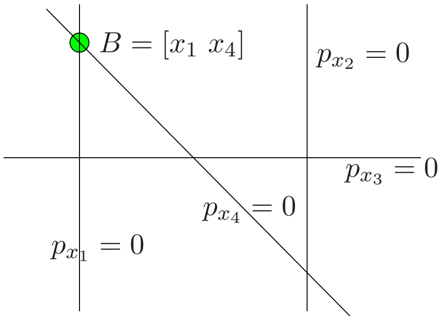
Corollary 1.3 *Given the vertex set* $V(X) := V(X, \lambda)$ *of the hyperplane arrangement* $H(X) := H(X, \lambda) := \{ H_x : x \in X \}$, *we have that:*

1. $\mathcal{D}(X) = \Pi(V(X))$.

2. *The restriction map* $p \mapsto p_{|V(X)}$ *is a bijection between* $\mathcal{P}(X)$ *and* $\mathbb{C}^{V(X)}$.

The *proof* is immediate: for the first assertion, we already proved that $\Pi(V(X)) \subset \mathcal{D}(X)$ (Lecture 3), and now we also know that $\dim \Pi(V(X)) = \#V(X) = \#\mathbb{B}(X) = \dim \mathcal{D}(X)$. The second assertion follows from the duality between $\mathcal{D}(X)$ and $\mathcal{P}(X)$ and the fact that $\mathcal{D}(X) = \Pi(V(X))$, by general principles. However, we already provided a direction proof for this in the discussion above. $\square$

Note that $V(X)$ depends on the selection of λ in the hyperplane arrangement, while $\mathcal{D}(X)$ depends only on X. The corollary can be extended to the non-generic case. That extension will be discussed in the next lecture.



$$X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = [x_1, x_2, x_3, x_4]$$

$$p_{X\backslash B} = p_{x_2} p_{x_3}$$

Figure 10. Example: the polynomial $p_{X\backslash B}$ vanishes on $V(X)\backslash v_B$.

$\square$

2 Notes on this lecture

The isomorphism between $\mathcal{D}(X)$ and $\mathcal{P}(X)$ was established in

N. Dyn, A. Ron, Local approximation by certain spaces of exponential polynomials, approximation order of exponential box splines and related interpolation problems, Transactions of Amer. Math. Soc. **319** *(1990), 381-404.*

The result there established the isomorphism between $\mathcal{P}(X)$ and a larger class of spaces that include $\mathcal{D}(X)$ as well as $\mathrm{Exp}(V(X))$ (for generic λ).

The cobasis basis for $\mathcal{P}(X)$ was also introduced in this paper. Indeed, the proof presented in this lecture is taken from that paper.