Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

Lectures given by Amos Ron
and recorded by Yeon Hyang Kim

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1 Ideals.

Let

$$\Pi := \mathbb{C}[t_1, \cdots, t_n].$$

Definition 1.1 *A subring $I \subset \Pi$ is an* **ideal** *if*

$$\Pi I := \{ \sum_{p \in \Pi, q \in I} pq \} = I.$$

□

Example 1.2 *Let $A := \{ p_1, \cdots, p_k \} \subset \Pi$. Define*

$$\ker A := \{ f \in \Pi \, : \, p_i(D) f = 0, \quad \forall i \} .$$

Then with

$$I_A := \left\{ \sum_{i=1}^{k} p_i q_i \, : \, q_i \in \Pi \right\}$$

the ideal generated by A (i.e., the smallest ideal that contains A), we have that $p(D) \ker A = 0$, for every $p \in I_A$. □

The passage from a finite set of differential operators to their ideal allows one to switch from differential operators to linear functionals, hence to employ, whenever needed and possible, duality arguments:

Theorem 1.3 *With A and I_A as above, define*

$$E_1 := \{\, \lambda \in \Pi \,:\, \lambda \in \ker A \,\},$$

$$E_2 := \{\, \lambda \in \Pi \,:\, p(D)\lambda = 0, \ \forall\, p \in I \,\},$$

$$E_3 := \{\, \lambda \in \Pi \,:\, \langle \lambda, p \rangle = 0, \ \forall\, p \in I \,\}.$$

Then

$$E_1 = E_2 = E_3.$$

□

Note that we have looked so far for *polynomials* that are annihilated by the differential operators in A (or I_A). In general those polynomials carry insufficient information about the ideal I_A (they do only in case all the associated primes in some primary decomposition of the ideal contain the constant 1 in their variety).

To this end, we extend the notion of $\ker I$ to include other functionals in Π'. Given an ideal I, we define

$$\ker I := \operatorname{span} \{\, e_\alpha p \,:\, \alpha \in \mathbb{C}^n, p \in \Pi \text{ s.t. } \langle e_\alpha p, q \rangle = 0, \quad \forall\, q \in I \,\}.$$

Recall that

$$e_\alpha : t \mapsto e^{\alpha \cdot t},$$

$$\langle e_\alpha p, q \rangle = q(D)(e_\alpha p)(0) = (p(D)q)(\alpha).$$

Definition 1.4 *The variety* $\mathrm{Var}(I)$ *of I is the set*

$$\{\, e_\alpha \,:\, e_\alpha \in \ker I \,\}.$$

□

Remark: The above definition of $\ker I$ suffices (i.e., there is no need to look further for more general linear functionals that are annihilated by I) since $\ker I$ is **total** in the sense that

$$(\ker I) \perp := \{\, q \in \Pi \,:\, \langle k, q \rangle = 0, \quad \forall\, k \in \ker I \,\} = I.$$

In contrast, $\mathrm{Var}(I)$ is not total, i.e., the ideal $\mathrm{Var}(I) \perp$, which is known as the *radical of I*, is usually larger than I.

Theorem 1.5 *Given $e_\alpha \in \mathrm{Var}(I)$, we define*

$$(\ker I)_\alpha := \{ e_\alpha p \,:\, e_\alpha p \in \ker I \} =: e_\alpha \Pi_\alpha,$$

and

$$I_\alpha := (\ker I)_\alpha \perp .$$

Then there exist a finite subset $(e_\alpha)_\alpha \subset \mathrm{Var}(I)$ such that the localized ideals $(I_\alpha)_\alpha$ determine I:

$$I = \cap_\alpha I_\alpha.$$

□

A *proof* of this result is found in the Appendix to this lecture.

Definition 1.6 *An ideal I is called 0-dimensional if its variety is finite.* □

We note that I is 0-dimensional if and only if Π/I is a finite dimensional, i.e., if I has finite codimension.

Discussion Assume that I is 0-dimensional. Then $\mathrm{Var}(I)$ is finite, and, moreover, the multiplicity spaces $(\ker I)_\alpha$ (for $e_\alpha \in \mathrm{Var}(I)$) are finite dimensional, too. In this case, $\ker I$ is *finite dimensional* space of exponential-polynomials:

$$\ker I := \sum_{e_\alpha \in \mathrm{Var}(I)} \ker I_\alpha,$$

and

$$\dim(\ker I) = \dim \Pi/I.$$

Theorem 1.7 *If I is a 0-dimensional ideal, then*

$$\Pi = I \oplus (\ker I)_\downarrow.$$

($F_\downarrow := \mathrm{span}\{ f_\downarrow \,:\, f \in F \}$.)

2 The dual action.

For a 0-dimensional ideal I, we define

$$I_{\uparrow} := \operatorname{span}\{\, p_{\uparrow} \ :\ p \in I \,\}.$$

$I_{\uparrow}$ is a *homogeneous* ideal, i.e., is generated by homogeneous polynomials. The variety of a 0-dimensional homogeneous ideal contains only the constant 1:

Theorem 2.1 *Given a* 0*-dimensional ideal* I*, we have the followings:*

1. $\operatorname{Var}(I_{\uparrow}) = \{\, e_0 \,\}$.

2. $\ker(I_{\uparrow}) = (\ker I)_{\downarrow}$.

3. $(I_{\uparrow}) \oplus \ker(I_{\uparrow}) = \Pi$.

□

3 Ideals over hyperplane arrangements

Let us connect now the general discussion with our specific context. As before, $X \subset \mathbb{R}^n\backslash 0$ is a multiset. Recall that each $x \in X$ is associated with a homogeneous and a non-homogeneous polynomials:

$$q_x : t \mapsto x \cdot t,$$

$$p_{x,\lambda} : t \mapsto x \cdot t - \lambda.$$

Let

$$J(X)$$

be the ideal generated by the long polynomials $\{\, q_Y \ :\ Y \in L(X) \,\}$.[1]

It is easy to check that the only common zero of the long polynomials is 0. Equivalently:

1. $\operatorname{Var}(J(X)) = \{\, e_0 \,\}$.

This observation tells us that the kernel of $J(X)$ is finite dimensional and is a subspace of Π. That is the reason that we defined $\mathcal{D}(X)$ as a subspace of Π:

2. $\ker J(X) = \mathcal{D}(X) \subset \Pi$.

[1]Why $J(X)$ and not $I(X)$? We reserve the latter notation to the dual ideal, i.e., the one whose kernel is $\mathcal{P}(X)$, and which will be introduced in the next lecture.

Let $J(X,\lambda)$ be the ideal generated by the *inhomogeneous* long polynomials:[2][3]

$$\{\, p_{Y,\lambda} := \Pi_{y\in Y} p_{y,\lambda} \;:\; Y \in L(X)\,\}\,.$$

It is easy to see that the variety of $J(X,\lambda)$ is the vertex set $V(X,\lambda)$ of the arrangement $H(X,\lambda)$ that is defined by the polynomials $(p_{x,\lambda} : x \in X)$.

One of the main challenges in the theory so far was to compute the dimension of $\mathcal{D}(X)$. In terms of the annihilating ideal $J(X)$ of $\mathcal{D}(X)$, our approach was as follows:

1. We "inhomogenized" the generators $\{q_Y\}$ of $J(X)$ (replaced q_Y by p_Y), to obtain the ideal $J(X,\lambda)$.

2. We computed the variety of $J(X,\lambda)$ and counted its cardinality. That gave us a lower bound on $\dim\ker J(X,\lambda)$.

3. We concluded that

$$\dim\ker(J(X,\lambda)_{\uparrow}) = \dim\ker J(X,\lambda) \geq \#\mathrm{Var}(J(X,\lambda)).$$

As asserted earlier, the homogenization $I \mapsto I_{\uparrow}$ is always stable, i.e., preserves the codimension (or, in other words, the dimension of the kernel). However, in general, the inhomogenization is not stable and may reduce kernel dimensions: there was not an automatic way to know that $J(X,\lambda)_{\uparrow} = J(X)$. Therefore, we had to use other arguments in order to move from the inequality $\dim\mathcal{D}(X) \geq \#\mathbb{B}(X)$ to the final equality. Ideal theory does not seem to deliver that latter part.

Note also that we focused on the generic perturbation. The reason was that we wanted to maximize the cardinality of the variety of the ideal $J(X,\lambda)$. In the generic case, we obtained $\#\mathbb{B}(X)$ exponentials in the variety. The upper bound $\dim\mathcal{D}(X) \leq \#\mathbb{B}(X)$ tells us that no lower order perturbation of the long polynomial $(q_Y)_Y$ can yield a larger variety.

So far, we focused on the one hand on the central arrangement ($\lambda = 0$), the homogeneous $J(X)$ and its kernel $\mathcal{D}(X)$, and on the other hand on the radical $J(X,\lambda)$ whose kernel is spanned by its variety. The non-central non-generic selection of λ leads to the study of the kernels

$$\mathcal{D}(X,\lambda) := \ker J(X,\lambda) = \mathrm{span}\,\{\, e_\alpha f \;:\; \alpha \in \mathbb{C}^n,\ f \in \Pi \text{ s.t. } p_{Y,\lambda}(D)(e_\alpha f) = 0,\ \forall\, Y \in L(X)\,\}\,.$$

We will briefly discuss those spaces in the next lecture. Then, we will switch direction and focus on $\mathcal{P}(X)$ and its association with the dual geometry of the zonotopes.

[2] We have just added the script λ to the previous notation p_x, in order to stress the dependence on λ

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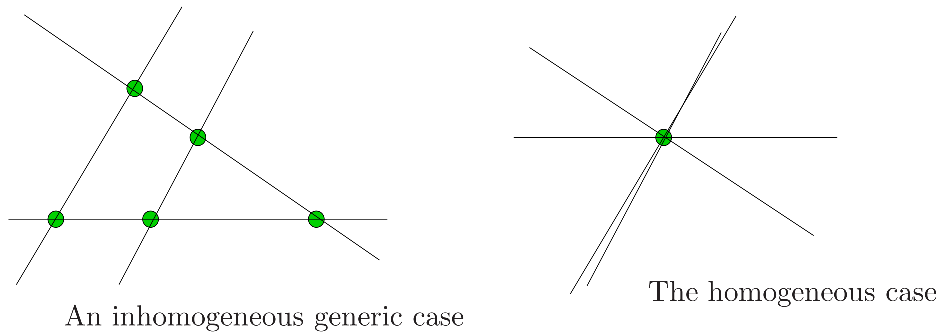

Figure 11. Varieties

Note that the generic case is related to the central case as follows:

$$\ker J(X,\lambda) = \operatorname{span}\{\, e_\alpha : \alpha \in V(X,\lambda)\,\}.$$

$$J(X,\lambda)_\uparrow = J(X).$$

$$(\ker J(X,\lambda))_\downarrow = \ker J(X) = \mathcal{D}(X).$$

4 Notes on this lecture

The majority of the results in this lecture are standard. Some of the actual presentation follows de Boor-Ron, JMAA, 1991. The space $\mathcal{D}(X)$ was introduced in de Boor - Höllig, JdAM, 1983. The spaces $\mathcal{D}(X,\lambda)$ were introduced in Ron, CA, 1988. The fact that kernels of polynomial ideals can be synthesized from finitely many localizations is found in Lefranc, CRAS, 1958, and the proof below follows Lefranc's sketch of proof. The backbone of that proof is the Noether-Lasker Theorem, which is stated and proved in the Appendix, too.[4]

5 Appendix

The discussion here assume some familiarity with polynomial ideals.

Theorem 5.1 *Let $I \subset \Pi$, and let $I = \cap Q_i$ be a primary decomposition of I. For each i, choose $e_{\alpha_i} \in \operatorname{Var} Q_i$. Then $K := \sum_i (\ker I)_{\alpha_i}$ is total for I:*

$$K \perp = I.$$

□

[4] "Noether" here is Max Noether, Emmy's father

Proof. We assume without loss that I is primary and that $e_0 \in \text{Var}(I)$. Set $K := (\ker I)_0$. The idea of the proof is straightforward: use Hahn-Banach. To this end, we realize Π' as the space $A = \mathbb{C}[[t_1, \ldots, t_n]]$ of formal power series. Then $I \subset \Pi \subset A$. We extend I to an ideal $J := IA$ in A. It is known that J is closed in the topology of A as a local ring (i.e., $(a_i)_i \subset A$ converges to 0 iff, for every positive integer k, almost all the elements in the sequence are orthogonal to polynomials of degree $\leq k$.) In that topology Π is the continuous dual of A. Thus,

$$J = \{a \in A : \langle a, p \rangle = 0, \ \forall p \in K\}.$$

Thus, the assertion of the theorem is proved once we know that $J \cap \Pi = I$, a property that is invalid in general but valid for primary ideals. □

Theorem 5.2 *With A, Π, I and J as above, $J \cap \Pi = I$, whenever I is primary.* □

Proof. Let R be the residue ring Π/I, and let M be the ideal of polynomials that vanish at 0. We first claim that (in R)

$$\cap_{j=1}^{\infty} (M/I)^j = 0. \tag{1}$$

Indeed, by Krull's Intersection Theorem x lies in this intersection if and only if it is the zero divisor of some element of the form $1 - m$, $m \in M/I$. So let us assume that for some x $x(1 - m) = q$, $m \in M$, $q \in I$. If $x \in I$, then $x = 0$ in R. Otherwise, since I is primary, a power of $1 - m$ lies in I, hence in M. But any power of $1 - m$ has the form $1 + m'$ where $m' \in M$, which yields $1 \in M$. A contradiction. Hence (1) is valid.

Now assume that p is a polynomial which is generated by I in the ring of formal power series. Then

$$p = \sum_j q_j a_j, \ q_j \in I, \ a_j \in A.$$

For each j and a positive integer m, let $a_{j,m}$ be a polynomial that satisfies $a_{j,m} - a_j \in M^m$. Define $p_m = \sum_j q_j a_{j,m}$. Note $p - p_m \in M^m$, while $p_m \in I$, which together implies that $p \in (M/I)^m$. By (1) we can now conclude that $p \in I$, as claimed. □