Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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Lecture 6: March 2, 2007

1 The multiplicities of the vertices of the hyperplane arrangement

Let $\lambda \in \mathbb{C}^X$ be given. Recall that we associate each $x \in X$ with a polynomial

$$p_{x,\lambda} : t \mapsto x \cdot t - \lambda_x.$$

Let $V(X, \lambda)$ be the vertices of the corresponding hyperplane arrangement. In general, the map

$$B \in \mathbb{B}(X) \mapsto v_B \in V(X, \lambda)$$

is not injective. (For example, if $\lambda = 0$ then $v_B = 0$ for every B.) To this end, we define

$$X_v := \{ x \in X \ : \ p_{x,\lambda}(v) = 0 \},$$

where $p_{x,\lambda}$ is the above inhomogeneous polynomial associated with x. We further define

$$J(X, \lambda)$$

to be the ideal generated by the polynomials

$$p_{Y,\lambda} := \prod_{y \in Y} p_{y,\lambda}, \quad Y \in L(X).$$

It is straightforward to check that

$$J(X) := J(X, 0) = J(X, \lambda)_{\uparrow},$$

hence that, with

$$\ker(J(X, \lambda)) := \text{span} \{ e_\alpha f \ : \ \alpha \in \mathbb{C}^n, f \in \Pi \text{ s.t. } p_{Y,\lambda}(D)(e_\alpha p) = 0, \ \forall Y \in L(X) \},$$

that

$$\mathcal{D}(X) = \ker J(X) = (\ker J(X,\lambda))_{\downarrow}.$$

In the generic case, when the map $v \to v_B$ is 1-1, we identified the ideal $J(X,\lambda)$ as the radical ideal with kernel $\mathrm{Exp}(V(X,\lambda))$. For a general λ, we define

$$\mathcal{D}(X,\lambda) := \oplus_{v\in V(X,\lambda)} e_v \mathcal{D}(X_v).$$

Note that every basis $B \in \mathbb{B}(X)$ lies in exactly one of the sets X_v, $v \in V(X,\lambda)$ (viz., in X_{v_B}), hence

$$\dim \mathcal{D}(X,\lambda) = \sum_{v\in V(X,\lambda)} \#\mathbb{B}(X_v) = \#\mathbb{B}(X).$$

On the other hand, it is easy to check that

$$\mathcal{D}(X,\lambda) \subset \ker(J(X,\lambda)),$$

and we further know that

$$\dim\ker(J(X,\lambda)) = \dim\ker(J(X,\lambda))_{\downarrow} = \dim\ker J(X) = \dim \mathcal{D}(X) = \#\mathbb{B}(X).$$

Hence we proved:

Theorem 1.1

$$\mathcal{D}(X,\lambda) = \ker J(X,\lambda).$$

□

Example 1.2 *For* $X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, and a suitable choice of λ, the hyperplane arrangement looks as below.

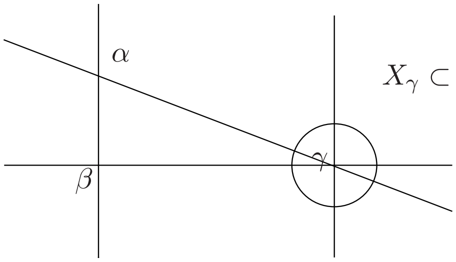

Figure 13. Local structure

In this case

$$\mathcal{D}(X,\lambda) = \operatorname{span}\{\, e_\alpha,\ e_\beta,\ e_\gamma \mathcal{D}(X_\gamma)\,\}.$$

The space $\mathcal{D}(X_\gamma)$ is easily found to be Π_1. Altogether,

$$\dim \mathcal{D}(X,\lambda) = 5 = \#\mathbb{B}(X).$$

The space $\mathcal{P}(X)$ is dual to each of the spaces $\mathcal{D}(X,\lambda)$, $\lambda \in \mathbb{C}^X$. This follows from a general principle.[1] Moreover, for each $\lambda \in \mathbb{C}^X$, $\mathcal{P}(X)$ admits a decomposition that is dual to the structure of $\mathcal{D}(X,\lambda)$:

$$\mathcal{P}(X) = \oplus_{v \in V(X,\lambda)}\, p_{X \backslash X_v,\lambda}\mathcal{P}(X_v).$$

2 Ideals over zonotopes.

We focus now on the space

$$\mathcal{P}(X) := \operatorname{span}\{\, p_Y \;:\; Y \in S(X)\,\}\,.$$

We will build a homogeneous basis for $\mathcal{P}(X)$, which will enable us to compute the homogeneous dimensions of $\mathcal{P}(X)$. Let us formalize this step: for $j = 0, 1, \cdots$, we define

$$h_X(j) := \dim(\mathcal{P}(X) \cap \Pi_j^0),$$

and refer to h_X as the **central Hilbert function of** X. The reference to h_X as "Hilbert function" is justified by the fact that it is the Hilbert function of the ideal $J(X)$ (or, more generally, each of the ideals $J(X,\lambda)$), and well as the Hilbert function of the forthcoming ideal $I(X)$ whose kernel is $\mathcal{P}(X)$. The adjective "central" was chosen in anticipation of the introduction of two other Hilbert functions that will be labeled "internal" and "external" respectively.

An algorithm for computing h_X.
First, we order X in some way. Then we associate each $B \in \mathbb{B}(X)$ with a homogeneous polynomial $Q_B := q_{Y_B}$, where

$$Y_B := \{\, y \in X \;:\; y \notin \operatorname{span}\{\, b \in B \;:\; b \leq y\,\}\,\}\,.$$

Note that $Y_B \in S(X)$, since $B \subset (X \backslash Y_B)$.

Example 2.1 *Let* $X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =: [x_1\ x_2\ x_3\ x_4]$ *be an ordered multiset. Then*

$$B = [x_1\ x_3] \Longrightarrow Y_B = \phi,$$

$$B = [x_2\ x_4] \Longrightarrow Y_B = \{\, x_1,\ x_3\,\}\,.$$

The algorithm easily produces the Hilbert function

$$h_X = (1, 2, 2, 0, 0, \ldots)$$

□

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[1] If F is a finite dimensional space of the power series space $\mathbb{C}[[t_1, \ldots, t_n]]$, and G is a homogeneous polynomial space that is dual to $F_\downarrow$, then it is also dual to F.

Theorem 2.2 *$\{ Q_B := q_{Y_B} \, : \, B \in \mathbb{B}(X) \}$ is a basis for $\mathcal{P}(X)$.*

Proof. It is clear that

$$Q_B \in \mathcal{P}(X),$$

since $Y_B \cap B = \emptyset$, i.e., $Y_B \in S(X)$. Since $\dim \mathcal{P}(X) = \#\mathbb{B}(X)$, it is sufficient to show that

$$\{ Q_B := q_{Y_B} \, : \, B \in \mathbb{B}(X) \} \tag{1}$$

is linearly independent. We will prove this by induction on $\#X$ and n. Assume the set in (1) to be linearly independent. Let $X' = X \cup \{ \xi \}$ where ξ is the last element in X'. The induction step requires us to show that, given $p \in \mathcal{P}(X)$ and $B_0 \in B \in \mathbb{B}(X') \backslash \mathbb{B}(X)$, if

$$p + \sum_{B \in \mathbb{B}(X') \backslash \mathbb{B}(X)} a(B) Q_B = 0,$$

then $a(B_0) = 0$. To this end, we define

$$B_0 =: B_0' \cup \{ \xi \}, \quad H := \operatorname{span} B_0'.$$

Let η be a non-zero vector such that $\eta \perp H$. Then

$$0 = D_\eta^{m(H)} 0 = D_\eta^{m(H)} p + \sum_{B \in \mathbb{B}(X') \backslash \mathbb{B}(X)} a(B) D_\eta^{m(H)} Q_B, \tag{2}$$

where $m(H) := \#(X \backslash H)$. Since $D_\eta q_Y = 0$ if $Y \subset H$,

$$D_\eta^{m(H)} q_Y = 0 \text{ unless } \#(Y \backslash H) \geq m(H).$$

Now, since every short set $Y \in S(X)$ cannot contain the entire set $X \backslash H$, we must have that $\#(Y \backslash H) < m(H)$, hence that $D_\eta^{m(H)} q_Y = 0$. Thus, $D_\eta^{m(H)} \mathcal{P}(X) = 0$, hence $D_\eta^{m(H)} p = 0$. Next, consider

$$D_\eta^{m(H)} Q_B, \quad B \in \mathbb{B}(X') \backslash \mathbb{B}(X).$$

We know that $\#(X' \backslash H) = m(H) + 1$. If $\#(B \cap H) < n - 1$, then $X' \backslash Y_B$ contains at least 2 vectors from $X' \backslash H$. So, $\#(Y_B \cap (X' \backslash H)) < m(H)$. Consequently,

$$D_\eta^{m(H)} Q_B = 0.$$

If $\#(B \cap H) = n - 1$, then $\#(Y_B \backslash H) = m(H)$ so that

$$D_\eta^{m(H)} Q_B = c_B q_{Y_B \cap H},$$

for some non-zero coefficient c_B. From equation (2), we get

$$0 = \sum_{B \in K} a(B) c_B q_{Y_B \cap H},$$

where $K := \{\, B \in \mathbb{B}(X') \backslash \mathbb{B}(X) \,:\, \#(Y_B \backslash H) = m(H) \,\}$. Now, it is easily observed that, with $Z := X \cap H$, the polynomials

$$q_{Y_B \cap H}, \quad B \in K$$

are exactly the polynomials in the homogeneous basis for $\mathcal{P}(Z)$, with the order on Z being the induced order from X. Our induction hypothesis on n implies that those polynomials are independent, hence that $a(B_0) = 0$. $\square$

3 Monomization of $\mathcal{P}(X)$

The valuation map val_X defined by

$$\mathbb{B}(X) \ni B \mapsto \# Y_B$$

depends on the order we choose, while the Hilbert function

$$h_X(j) = \# val_X^{-1}(j)$$

is independent of the chosen order. Since the order of X encodes the geometry of the hyperplane arrangement[2], it is useful to look for valuations that are sensitive to the order, too. This motivates the introduction of the **geometric valuation** *gval* of X:

$$gval : \mathbb{B}(X) \to \mathbb{Z}_+^d, \quad gval(B)(i) := \#\{y \in Y_B : \; y < b_i\} - gval(B)(i-1),$$

with b_i the ith vector in B (in the order of X), and $gval(B)(0) := 0$. The geometric valuation can be easily shown to be injective, hence leads to the monomial space

$$\mathcal{M}(X) := \operatorname{span}\{t^{gval(B)} : \; B \in \mathbb{B}(X)\},$$

where $t^\alpha := \prod_{i=1}^n t_i^{\alpha_i}$. The study of the geometric valuation and its connection to hyperplane arrangements and zonotope tilings is beyond the scope of these notes.

4 Notes on this lecture

The discussion on the structure of $\mathcal{D}(X, \lambda)$ is taken from Ben-Artzi, Ron, TAMS, 1988, with the fact that $\mathcal{D}(X, \lambda)_\downarrow = \mathcal{D}(X)$ being established in de Boor, Ron, JMAA, 1991. The homogeneous basis for $\mathcal{P}(X)$ was constructed in Dyn, Ron, TAMS, 1990.

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[2] There is slight inaccuracy in this statement. The geometry of the hyperplane arrangement is determined by the order of X and the sign of each vector $x \in X$.