Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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1 The annihilating ideal of $\mathcal{P}(X)$

The set of dual hyperplanes of X is defined as follows:

$$\mathcal{H}^*(X) := \{ H : H \text{ is a subspace of } \mathbb{R}^n,\ \dim H = n-1,\ \operatorname{span}(X \cap H) = H \}.$$

Given any dual hyperplane $H \in \mathcal{H}^*(X)$, let η_H be a non-zero normal to H: $\eta_H \perp H$. We also define

$$m(H) := m_X(H) = \#(X \backslash H).$$

It is easy to argue that

$$D_{\eta_H}^{m(H)}(\mathcal{P}(X)) = 0, \quad \forall H \in \mathcal{H}^*(X). \tag{1}$$

Define

$$I(X) := \operatorname{Ideal}\left\{ q_{\eta_H}^{m(H)} : H \in \mathcal{H}^*(X) \right\},$$

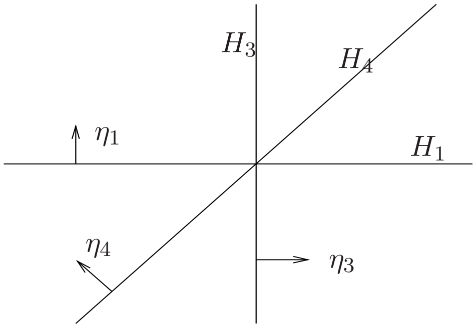
where $q_x : t \mapsto x \cdot t$. Then, by (1), $\mathcal{P}(X) \subset \ker I(X)$. Moreover, we have the following theorem:

Theorem 1.1

$$\mathcal{P}(X) = \ker I(X).$$

Example 1.2

$$X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =: [x_1, x_2, x_3, x_4].$$



$$\mathcal{H}^*(X) = \{\, H_i = \operatorname{span} x_i \,:\, i = 1, 3, 4 \,\},$$

$$I(X) = \operatorname{Ideal}\{\, q_{\eta_1}^2,\ q_{\eta_3}^3,\ q_{\eta_4}^3 \,\},$$

where $\eta_i := \eta_{H_i}$

Figure 14. $\mathcal{H}^*(X)$ and $I(X)$ $\square$

The *proof* of the above theorem (or, more, precisely the one that I know) goes by induction on $\#X$ and n. Since we know that $\mathcal{P}(X) \subset \ker I(X)$, and since we know that $\dim \mathcal{P}(X) = \#\mathbb{B}(X)$, it suffices to prove that

$$\dim \ker I(X) \le \#\mathbb{B}(X).$$

Assuming that this statement is correct for X, we define $X' := X \cup \{\xi\}$, and consider for every dual hyperplane $H \in \mathcal{H}^*(X)$ the space $P_H := \mathcal{P}((X \cap H) \cup \{\xi\})$. If $\xi \in H$, then $P_H = 0$; otherwise, P_H has positive dimension. Note that each $B \in \mathbb{B}(X')$ lies either in $\mathbb{B}(X)$ (in case it does not contain ξ), or else in a unique $(X \cap H) \cup \{\xi\}$, $H \in \mathcal{H}^*(X)$. Therefore,

$$\dim \mathcal{P}(X) + \sum_{H \in \mathcal{H}^*(X)} \dim \mathcal{P}_H = \#\mathbb{B}(X').$$

We then define a map T as follows:

$$T : \ker I(X') \to \prod_{H \in \mathcal{H}^*(X)} P_H,$$

by

$$f \mapsto (D_{\eta_H}^{m(H)} f)_H.$$

The kernel of this map is, by definition, $\ker I(X)$, hence, by induction, $\mathcal{P}(X)$. Our previous computation then shows that

$$\dim \ker I(X') \le \dim \ker T + \dim \operatorname{ran} T = \#\mathbb{B}(X').$$

The only missing item in the argument is to show that the map T is well-defined, i.e., that $D_{\eta_H}^{m(H)} \ker I(X') \subset P_H$. This is trivially true in case $\xi \in H$. Proving the above for the case $\xi \in H$ is the hard part of the proof, and is omitted. $\square$

2 Zonotopes

Recall the following definition:

Definition 2.1 *We say that X is* **unimodular** *if*

1. $X \subset \mathbb{Z}^n$*:*

2. $\forall B \in \mathbb{B}(X),\ B\mathbb{Z}^n = \mathbb{Z}^n\ (\iff\ det(B) = \pm 1)$. □

Also, the zonotope $Z(X)$ is defined as

$$Z(X) := X([0,1]^X),$$

where $X : \mathbb{R}^X \to \mathbb{R}^n$.

In the context of the hyperplane arrangement, a set of interest is the vertex set $V(X,\lambda)$ of the arrangement, whose precise geometry depends of the vector $\lambda \in \mathbb{C}^X$. For a generic λ, the vertex set $V(X,\lambda)$ is of maximal cardinality $\#\mathbb{B}(X)$. The dual vertex set, $V^*(X,t)$, is parameterized by $t \in \mathbb{R}^n$. As we will shortly see, for a generic t this set is of minimal cardinality $\#\mathbb{B}(X)$. Let's start with the definition:

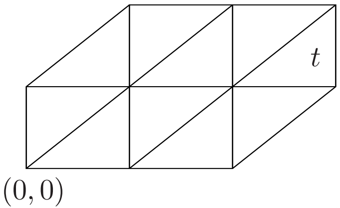
$$V^*(X,t) := \{\alpha \in \mathbb{Z}^n :\ t - \alpha \in Z(X)\}.$$

We consider t to be *generic* here if it does not lie in any of the hyperplanes

$$\alpha + H, \quad H \in \mathcal{H}^*(X),\ \alpha \in \mathbb{Z}^n.$$

Example 2.2

$$X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$



$V^*(X,t)$
$= \{\,(0,0),\ (1,0),\ (2,0),\ (1,1),\ (2,1)\,\}$

Figure 15. $V^*(X,t)$ □

Fact: If t is generic, then

$$\#V^*(X,t) = \mathrm{Vol}(Z(X)).$$

Corollary 2.3 *If X is unimodular and $t \in \mathbb{R}^n$ is generic, then*

$$\#V^*(X,t) = \mathrm{Vol}(Z(X)) = \#\mathbb{B}(X).$$

$\square$

Recall that for a given $\sigma \subset \mathbb{R}^n$, we defined

$$\mathrm{Exp}(\sigma)_{\downarrow} =: \Pi(\sigma).$$

Since we always have that $\dim \Pi(\sigma) = \#\sigma$, then, for a unimodular X and generic t,

$$\Pi(V^*(X,t)) = \#V^*(X,t) = \#\mathbb{B}(X).$$

Theorem 2.4 *If X is unimodular and $t \in \mathbb{R}^n$ is generic, then*

$$\mathcal{P}(X) = \Pi(V^*(X,t)).$$

Proof. Since both spaces in question have the same dimension, we only need to prove that one is included in the other. We will show that $\mathcal{P}(X) \supset \Pi(V^*(X,t))$.
Since we know that $\mathcal{P}(X) = \ker I(X)$, we are entitled to show that $\ker I(X) \supset \Pi(V^*(X,t))$.

To this end, we recall that, for $p, q \in \Pi$ with $p_{\uparrow} = q$, if $p|_{\sigma} = 0$, then

$$q(D)(\Pi(\sigma)) = 0.$$

We choose q to be one of the generators of $I(X)$, i.e.,

$$q : t \mapsto (\eta_H \cdot t)^{m(H)} = q_{\eta_H}^{m(H)}(t), \quad H \in \mathcal{H}^*(X).$$

We need to find p such that $p_{\uparrow} = q$ and p vanishes on $V^*(X,t)$. Once we manage to do so for every q as above, we are done. To this end, we fix $H \in H^*(X)$, and attempt to define p as (with $\eta := \eta_H$) $p := (q_\eta + c_1)(q_\eta + c_2)\cdots(q_\eta + c_{m(H)})$, with $(c_i)_i$ some constants. Obviously, for such p we always have that $p_{\uparrow} = q$. We need also to ensure that p vanishes on $V^*(X,t)$. In the argument below we assume for convenience that all vectors $X \backslash H =: \{y_1, \ldots, y_{m(H)}\}$ lie on one side of H. It then straightforward to see that the zonotope $Z(X)$ lies between the hyperplanne H, and the hyperplane

$$H' := H + \sum_{i=1}^{m(H)} y_i.$$

A simple consequence of the unimodularity is that there are exactly $m(H)-1$ translates of H that lies properly between H and H' and contain integers. Precisely, these are the hyperplanes

$$H + \sum_{i=1}^{j} y_i =: H + c_j, \quad j = 1, \ldots, m(H)-1.$$

Since t is generic, it does not lie on any of these hyperplanes, hence we may assume without loss that it lies between H and $H + y_1$. Then, we conclude that

$$V^*(X,t) \subset \cup_{j=0}^{m(H)-1} H + c_j, \quad c_0 := 0,$$

hence that the polynomial

$$p := \prod_{j=0}^{m(H)-1} (q_\eta + c_j)$$

vanishes on $\Pi(V^*(X,t))$. $\square$

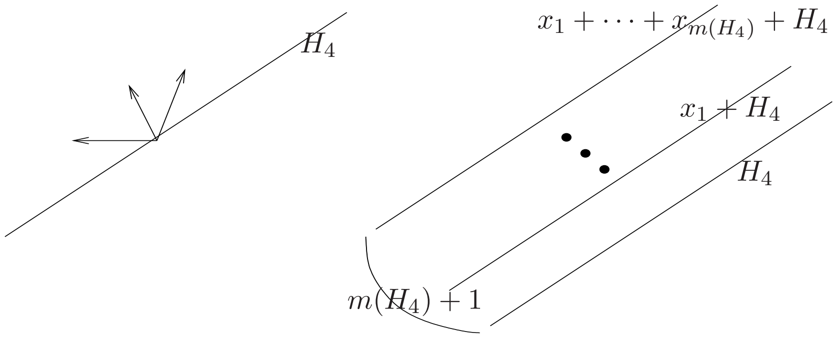


Figure 16. A geometric visualization of the above argument. We chose $H := H_4$

Corollary 2.5 *For a generic t and unimodular X, each of the spaces $\mathcal{P}(X)$ and $\mathcal{D}(X)$ interpolates correctly on the dual vertex set $V^*(X,t)$; i.e., the restriction map*

$$f \mapsto f|_{V^*(X,t)}$$

is a bijection between $\mathbb{C}^{V^*(X,t)}$ and each of the spaces $\mathcal{P}(X)$ and $\mathcal{D}(X)$.

3 Notes on this lecture

The dual vertex set $V^*(X,t)$ appears in Dahmen, Micchelli, Studia Math, 1984. The fact that $\mathcal{D}(X)$ interpolates correctly on $V^*(X,t)$ was proved there (using very different arguments than those presented here). The rest of the material in this lecture can be found in de Boor, Dyn, Ron, PJM, 1991. That latter reference contains the complete proof of the equality $\mathcal{P}(X) = \ker I(X)$.