Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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## 1 Tree inversions

Let $e_0 := 0$. Let $(e_i)_{i=1}^n$ be the standard basis for $\mathbb{R}^n$. Let $G$ be a connected undirectional graph with $n+1$ vertices and no loops (but with possibly multiple edges). We index the vertices by $[0:n]$, identify each vertex $i \in [0:n]$ with the vector $e_i$, and identify each edge connecting $i$ to $j$, $i<j$, with the vector $e_i - e_j$. In this spirit, we say that $X$ is **graphical** if

$$x \in \{\, e_i - e_j \,:\, 0 \le i,j \le j \,\}, \quad \forall x \in X.$$

Clearly, every graphical $X$ is associated with a graph, and vice versa (the only somewhat arbitrary choice here is the sign of the vector $e_i - e_j$). If each $e_i - e_j$, $i<j$, occurs exactly once in $X$, we refer to $X$ as *a complete graph*, since then the associated graph is so. If $X$ is graphical, we will use $X$ to denote the associated graph, too, following the tradition in graph theory to identify a graph with its set of edges.

There is a natural (and automatic) bijection between $\mathbb{B}(X)$ and the spanning trees in the graph $X$: one only needs to observe that the edges of $Y \subset X$ correspond to a spanning tree of $X$ if and only if $Y \in \mathbb{B}(X)$.

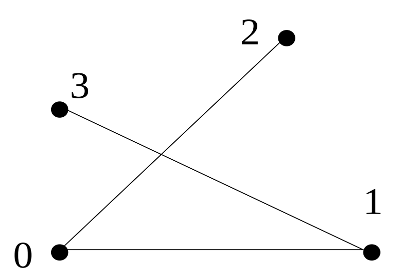


**Figure 17.** A spanning tree for $n = 3$

Recall that a subgraph of a spanning tree is called **forest**. Thus, there is also a natural bijection between the forests of the graph $X$ and the independent set $\mathbb{I}(X)$.

For each tree, we count the number of consistencies $\mathrm{con}(T)$ in the tree as follows: Choose an edge $(i,j) \subset [1:n]$, $i<j$. Then there is a unique path that connects 0 to $i$, using the edges of $T$. If $j$ does not lie on that path, we say that a pair $(i,j)$ is consistent. Otherwise, the edge $(i,j)$ is an **inversion**. Note that all the edges in Figure 17 are consistent. However, if we switch between the vertices 1 and 3, then the edge $(1,3)$ is an inversion. We define $\mathrm{con}(T)$ to be the number of consistent edges in the tree $T$.

**Theorem 1.1** *Let $X$ be a complete graph. Given any $0 \le k \le \binom{n}{2}$, we have*

$$\#\{\, T : T \text{ is a spanning tree such that } \mathrm{con}(T) = k \,\} = \dim(\Pi_k^0 \cap \mathcal{P}(X)) = h_X(k),$$

*where $h_X$ is the Hilbert function associated with $X$ (See lecture 6).* □

## 2 Parking functions

Let

$$s := [s_1, \cdots, s_n], \quad s_i \ge 0,\ i = 1, \cdots n,$$

be a sequence of $n$ non-negative numbers. If for each $i = 1, \cdots n$,

$$\#\{\, j : s_j < i \,\} \ge i,$$

then we say that $[s_1, \cdots, s_n]$ is a **parking function.**

**Example 2.1**

$$i = 1 \Longrightarrow \#\{\, j : s_j < 1 \,\} \ge 1.$$

So, there is at least one 0 in the sequence.

$$i = n \Longrightarrow \#\{\, j : s_j < n \,\} \ge n,$$

hence $s_j < n$, for all $j$.

For $n = 2$, $[0,0]$, $[0,1]$, $[1,0]$ are the only three parking functions, while, for $n = 3$, there are 16 parking functions which are obtained by permuting the following five:

$$[0,1,2],\ [0,0,1],\ [0,1,1],\ [0,0,2],\ [0,0,0].$$

Given a parking function $s := [s_1, \cdots, s_n]$, we define

$$|s| := s_1 + \cdots + s_n.$$

**Theorem 2.2** *Let $X$ be a complete graph with $n+1$ vertices. Then*

$$\#\{\text{ parking functions } s \text{ with } n \text{ digits } : |s| = k\,\} = \dim(\Pi_k^0 \cap \mathcal{P}(X)).$$

□

# 3 External theory

We begin now the discussion of the second part of the theory. In the first part, we focused on the set $\mathbb{B}(X)$ of the bases in $X$. In this part, we focus on the set

$$\mathbb{I}(X)$$

of independent sets in $X$. Our analysis here makes constant use of the idea of identifying $\mathbb{I}(X)$ as a subset of $\mathbb{B}(X')$, with

$$X' := X \cup B_0,$$

with $B_0$ some fixed (but arbitrary) basis for $\mathbb{R}^n$, whose vectors are ordered in some fixed way (see Lecture 1). The extension map

$$ex : \mathbb{I}(X) \to \mathbb{B}_+(X) \subset \mathbb{B}(X')$$

is done by completing $I$ to a basis in a greedy way from $B_0$; i.e., $b \in ex(I)$ iff ($b \in I$ or else ($b \in B_0$ and

$$b \notin \operatorname{span}\{I \cup \{b' \in B_0 : b' < b\}\}.))$$

That creates a 1-1 map from $\mathbb{I}(X)$ into $\mathbb{B}(X')$ whose range was denoted by $\mathbb{B}_+(X)$.

The external theory parallels the central one: we will introduce two spaces: one of the $\mathcal{P}$-type and one of the $\mathcal{D}$-type, identify their annihilating kernels, establish their duality, and connect the former to the vertices of the zonotope and the later to the connected components of the hyperplane arrangement. The first polynomial space, $\mathcal{P}_+(X)$, is defined as follows:

**Definition 3.1**

$$\mathcal{P}_+(X) := \operatorname{span}\{\, q_Y \;:\; Y \subset X\,\}\,.$$

While $\mathcal{P}_+(X)$ is obviously independent of the choice of the additional basis $B_0$, its counterpart $\mathcal{D}_+(X)$ does depend on this choice (as well as on the ordering of the vectors in $B_0$). We first define the following set of **externally long subsets of** $X$ (that are actually subsets of $X'$...):

$$L_+(X) := \{Y \subset X' :\; Y \cap B \neq \emptyset,\; \forall B \in \mathbb{B}_+(X)\}.$$

**Definition 3.2**

$$\mathcal{D}_+(X) := \{ f \in \Pi \,:\, q_Y(D)f = 0,\ \forall Y \in L_+(X) \} .$$

□

Note that $\mathcal{P}(X) \subset \mathcal{P}_+(X)$, and $\mathcal{D}(X) \subset \mathcal{D}_+(X)$. In Lecture 2, we defined, for any $\mathbb{B}' \subset \mathbb{B}(X')$,

$$\mathcal{D}_{\mathbb{B}'}(X') := \left\{ f \in \Pi \,:\, q_Y(D)f = 0, \quad \forall Y \subset X \text{ such that } Y \cap B \neq \emptyset,\ \forall B \in \mathbb{B}' \right\} .$$

We proved there that we always have that

$$\dim \mathcal{D}_{\mathbb{B}'}(X') \geq \#\mathbb{B}'(X').$$

Applying this general result to our particular case, we obtain:

**Lemma 3.3**

$$\dim \mathcal{D}_+(X) \geq \#\mathbb{B}_+(X) = \#\mathbb{I}(X).$$

Undoubtedly, our reader expects us to establish *equality*, as we will (but not in this lecture). In the meanwhile, here are the definitions of the associated ideals:

# 4 The two ideals in the external theory

The annihilating ideal

$$J_+(X)$$

of $\mathcal{D}_+(X)$ depends, necessarily, on $B_0$, and is defined as the ideal generated by the externally long polynomials:

$$J_+(X) := \text{Ideal}\,\{ q_Y \,:\, Y \in L_+(X) \} .$$

It is clear that $J_+(X) \subset J(X)$. The fact that

$$\mathcal{D}_+(X) = \ker J_+(X)$$

follows from the definition of the two spaces. In contrast, the annihilating ideal of $\mathcal{P}_+(X)$ is far less obvious: we define

$$I_+(X)$$

be the ideal generated by $\left\{ q_{\eta_H}^{m(H)+1},\ H \in \mathcal{H}^*(X) \right\}$. Note the pleasing connection between $I_+(X)$ and $I(X)$: we have just raised by 1 the power in each of the generators of the good old $I(X)$. We will prove that

$$\mathcal{P}_+(X) = \ker I_+(X),$$

that

$$\dim \mathcal{P}_+(X) = \dim \mathcal{D}_+(X) = \#\mathbb{I}(X),$$

and a few other results.

## 5 Notes on this lecture

The connection between graph inversions, parking functions and the central Hilbert function follows from the results of Gessel and Sagan, EJC, 1996 (One needs to this end to compare the definition of external activity in graph theory to our algorithm for constructing a basis for $\mathcal{P}(X)$). The connection extends to general graphs (Postnikov and Shapiro, TAMS, 2004).

The external theory in its entirety, as well as the internal one (yet to come), will be the content of a forthcoming paper of Olga Holtz and mine. Thus, the entire discussion from this point on can be viewed as an *extended announcement* of those results.