Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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1 External theory

Let $\mathbb{I}(X)$ be the independent set and $B_0 \subset \mathbb{R}^n$ be a fixed ordered basis. We denoted $X' := X \cup B_0$ and defined bijection $\mathrm{ex} : \mathbb{I}(X) \to \mathbb{B}_+(X) \subset \mathbb{I}(X)$. We also define d

$$L_+(X) := \left\{ Y \subset X' \,:\, Y \cap B \neq \phi, \quad B \in \mathbb{B}_+(X) \right\},$$

$$\mathcal{P}_+(X) := \operatorname{span}\{\, q_Y \,:\, Y \subset X \,\},$$

$$\mathcal{D}_+(X) := \{\, f \in \Pi \,:\, q_Y(D)f = 0, \ \forall Y \in L_+(X) \,\}.$$

We want to show now that $\mathcal{D}_+(X)$ and $\mathcal{P}_+(X)$ are dual to each other, and to determine their annihilating ideals. The result will be established in four steps.

1. As observed last time, we have that

$$\dim \mathcal{D}_+(X) \geq \#\mathbb{B}_+(X) = \#\mathbb{I}(X).$$

2. We will show that the map

$$\mathcal{D}_+(X) \to \mathcal{P}_+(X)' \,:\, p \mapsto \langle p, \cdot \rangle, \quad \langle p, q \rangle := p(D)q(0)$$

is injective.

3. Define

$$I_+(X) := \operatorname{Ideal}\left\{ q_{\eta_H}^{m(H)+1} \,:\, H \in \mathcal{H}^*(X) \right\},$$

where $\mathcal{H}^*(X)$ is the set of dual hyperplanes, $\eta_H \perp H, \quad m(H) := \#(X \backslash H)$. Then, we check directly that

$$\mathcal{P}_+(X) \subset \ker I_+(X).$$

(Indeed, given any $Y \subset X$ and any dual hyperplane H, we have that $D_{\eta_H}(q_Y) = q_{Y \cap H} D_{\eta_H} q_{Y \backslash H}$. The result then follows from the fact that $\#(Y \backslash H) \leq m(H)$.)

4.
$$\dim \ker I_+(X) \le \#\mathbb{B}_+(X).$$

Proof of 2. Define
$$J_+(X) := \mathrm{Ideal}\,\{\, q_Y \,:\, Y \in L_+(X) \,\}\,.$$
Note that $\ker J_+(X) = \mathcal{D}_+(X)$. We will show that
$$\mathcal{P}_+(X) + J_+(X) = \Pi,$$
which is equivalent to our claim.[1]

The proof itself starts with the fact that $\mathcal{P}(X) + J(X) = \Pi$ (that follows from the duality between $\mathcal{P}(X)$ and $\mathcal{D}(X)$). Since $\mathcal{P}_+(X) \supset \mathcal{P}(X)$, we conclude that
$$\mathcal{P}_+(X) + J(X) = \Pi.$$
So, we need to prove
$$J(X) \subset \mathcal{P}_+(X) + J_+(X).$$
Let $Y \in L(X)$, $f \in \Pi$. Since every polynomial in $J(X)$ is a combination of polynomials of the form $q_Y f$, it suffices to prove that
$$q_Y f \in \mathcal{P}_+(X) + J_+(X),$$
a claim that we prove by reverse induction on $\#Y$. Thus, assume that the claim is correct for every $Y' \in L(X)$ such that $\#Y' > \#Y$. Put $S := \mathrm{span}(X \backslash Y)$. Then $\dim S < n$, since Y is long. Let $I \subset X \backslash Y$ be a basis for S and $B := \mathrm{ex}(I)$. Since B is a basis for $\mathbb{R}^n$,
$$\mathrm{Ideal}\,\{\, q_b \,:\, b \in B \,\} + \Pi_0^0 = \Pi.$$
So, we can write f in the following form:
$$f = c_0 + \sum_{b \in B} q_b f_b, \quad f_b \in \Pi.$$
Consequently,
$$q_Y f = c_0 q_Y + \sum_{b \in B} q_{Y \cup \{\, b \,\}} f_b.$$
We claim that each term above belongs to $\mathcal{P}_+(X) + J_+(X)$. Since $Y \subset X$, it is clear that $q_Y \in \mathcal{P}_+(X)$. Now, for $q_{Y \cup \{\, b \,\}}$, we have either $b \in I$ or $b \in B_0$.

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[1] Indeed, assume that $p \in \mathcal{D}_+(X)$ and $p \perp \mathcal{P}_+(X)$. Since $\mathcal{D}_+(X) \perp J_+(X)$, we have that $p \perp \mathcal{P}_+(X) + J_+(X) = \Pi$, which implies $p = 0$. The converse is obtained by a similar argument.

Case I. If $b \in I \subset X$, then $Y' := Y \cup \{ b \} \subset X$. By induction,

$$q_{Y \cup \{ b \}} f_b \in \mathcal{P}_+(X) + J_+(X).$$

Case II. Let $b \in B_0$. We show that

$$q_{Y \cup \{ b \}} f_b \in J_+(X),$$

and to this end it is enough to show that $Y \cup \{ b \} \in L_+(X)$. Let $B' \in \mathbb{B}_+(X)$. If $Y \cap B' = \emptyset$, then $B' \cap X \subset X \backslash Y \subset S$. Hence $B' = ex(I')$, for $I' \subset S$. Thus $\mathrm{span} I' \subset \mathrm{span} I$, and the definition of the extension map implies that in such case we always have that $B_0 \cap \mathrm{ex}(I) \subset B_0 \cap \mathrm{ex}(I')$. Consequently, $b \in B'$, and thus

$$(Y \cup \{ b \}) \cap B' \neq \emptyset.$$

We conclude that $Y \cup \{ b \} \in L_+(X)$, hence that, directly from the definition of $J_+(X)$, $q_{Y \cup \{ b \}} f_b \in J_+(X)$; *a fortiori* $q_{Y \cup \{ b \}} f_b \in \mathcal{P}_+(X) + J_+(X)$.

□

Note that the only property of *ex* that we used in the proof was that, once $\mathrm{span} I' \subset \mathrm{span} I$ then $B_0 \cap I \subset B_0 \cap I'$. It is probably easy to show that every extension of such type is a greedy extension with respect to some ordering of B_0.

Corollary 1.1

$$\dim \mathcal{P}_+(X) \geq \mathcal{D}_+(X).$$

It follows then that $\dim \mathcal{P}_+(X) \geq \#\mathbb{I}(X)$. This last estimate can be proved directly: order X' such that B_0 is placed after X, and the internal order within B_0 is retained. Then follow the construction of a homogeneous basis for $\mathcal{P}(X')$ (Lecture 6). Observe that a polynomial Q_B, $B \in \mathbb{B}(X')$, in that basis is of the form q_{Y_B}, $Y_B \subset X'$, and that Y_B is then a subset of X if (and only if) $B \in \mathbb{B}_+(X)$. Thus

$$\{ Q_B : \ B \in \mathbb{B}_+(X) \} \subset \mathcal{P}_+(X),$$

and we get the desired bound from the linear independence of these polynomials. We will come back to this issue later, since the polynomials above form a *basis* for $\mathcal{P}_+(X)$, and we will use the cardinality of the sets Y_B, $B \in \mathbb{B}_+(X)$ in order to provide an algorithm for computing the forthcoming external Hilbert function h_X^+ of X.

2 Notes on this lecture

The results are taken from Holtz, Ron, 2007. The proof here for the embedding of $\mathcal{D}_+(X)$ in $\mathcal{P}_+(X)'$ looks (at least to me) like best possible. At a minimum, it is much better than our original argument (from the previous millennium, but unpublished) that went by induction on the spatial dimension.