Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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The topic will bring together aspects of linear algebra, convex geometry, combinatorics, commutative algebra, and analysis.

1 Linear Algebra.

Our entire investigation concerns a finite multiset $X \subset \mathbb{R}^n \backslash \{0\}$ of full rank n. At times, we will associate X with some (full) ordering. In this case, we may consider the vectors in X to comprise the columns of an $n \times X$ matrix, which we will still denote by X.

The set of all bases of X is denoted by $\mathbb{B}(X)$:

$$\mathbb{B}(X) := \{ B \subset X : B \text{ is a basis for } \mathbb{R}^n \}.$$

Example 1.1 Let X be the following multiset of four vectors $x_1, \ldots, x_4$ in $\mathbb{R}^2 \backslash \{0\}$:

$$X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = [x_1, x_2, x_3, x_4].$$

Note that we have here 5 bases. To visualize a basis $B \in \mathbb{B}(X)$, we consider B as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$, and choose the parallelepiped $B([0,1]^n)$ as our visualization of B. In this example we have the following 5 bases:

$$[x_1, x_3],\ [x_2, x_3],\ [x_1, x_4],\ [x_2, x_4],\ [x_3, x_4],$$

and they correspond to the following 5 parallelepipeds:

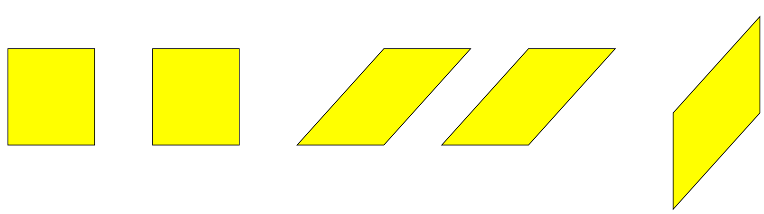


Figure 1: The five parallelograms of Example 1.1

□

We next define

$$\mathbb{I}(X) := \{\, I \subset X,\ \ I \text{ is independent in } \mathbb{R}^n \,\}\,.$$

Note that the empty set is defined to be independent. Thus, in Example 1.1, we have $\#\mathbb{I}(X) = 10$.

It is sometimes convenient to consider the independent sets as full-rank bases, too. To this end, we choose a fixed basis B_0 of $\mathbb{R}^n$, and append B_0 to X:

$$X' := X \cup B_0.$$

We then impose some arbitrary, but fixed, ordering on B_0, and associate each $I \in \mathbb{I}(X)$ with $B_I \in \mathbb{B}(X')$ which is the greedy completion of I to a basis, using the elements of B_0; i.e., $b \in B_I$ iff ($b \in I$ or else ($b \in B_0$ and

$$b \notin \operatorname{span}\{I \cup \{b' \in B_0 : b' < b\}\}.))$$

That creates a 1-1 map from $\mathbb{I}(X)$ into $\mathbb{B}(X')$. We will denote

$$\mathbb{B}_+(X) := \left\{\, B_I \in \mathbb{B}(X')\ :\ \ I \in \mathbb{I}(X) \,\right\}.$$

We refer to the bases in $\mathbb{B}_+(X)$ as the *external bases* of X. Note that every basis of X is external, by definition, and that not every external basis of X is a basis of X.

Next, we define the notion of an *internal basis*. To this end, we order X in an arbitrary way. Then $B \in \mathbb{B}(X)$ is said to be an **internal basis** if for each $b \in B$, b is not the last element in $X \backslash H$, where $H := \operatorname{span}(B \backslash b)$. We denote

$$\mathbb{B}_-(X) := \{\, B \in \mathbb{B}(X)\ :\ B \text{ is an internal basis } \,\}\,.$$

Example 1.2 Suppose X is ordered in the way given by Example 1. Then

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \notin \mathbb{B}_-(X), \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{B}_-(X).$$

In fact,

$$\mathbb{B}_-\left(\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right) = \{\, [x_1, x_3]\,,\ [x_2, x_3] \,\}\,.$$

If we change the order in X,

$$\mathbb{B}_-\left(\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}\right) = \{\, [x_1, x_2]\,,\ [x_2, x_3] \,\}\,.$$

□

It is obvious that the notion of an internal basis depends on the ordering. We claim, however, (so far without proof) that the number of internal bases is independent of the order of X.

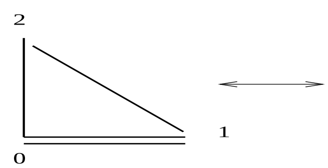
Definition 1.3 *We say that X is* **unimodular** *if*

1. $X \subset \mathbb{Z}^n$*:*

2. $\forall B \in \mathbb{B}(X),\ span_{\mathbb{Z}} B = \mathbb{Z}^n\ (\iff det(B) = \pm 1)$. □

From now on, we assume that X is unimodular. Many of the results presented here extend to non-unimodular setups.

Example 1.4 The edge set of a graph G.
Let G be a connected undirectional graph with $n+1$ vertices. Let $e_0 := 0$. Let $(e_i)_{i=1}^n$ be the standard basis for $\mathbb{R}^n$. An edge e_{ij} the connects the vertices i and j is associated with the vector $e_i - e_j \in \mathbb{R}^n$. We then choose X to be the "edge set" of G. For example, the graph on the left is associated with the multiset on the right:



$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Figure 2: Graph and X

□

Note that the edge (multi)set X of a graph is always unimodular.

2 Hyperplane arrangements.

We first associate each $x \in X$ with a constant $\lambda_x \in \mathbb{R}$, and define a linear polynomial p_x:

$$p_x : \mathbb{R}^n \to \mathbb{R} : t \mapsto x \cdot t - \lambda_x.$$

The hyperplane arrangement of X is the arrangement induced by the zero sets of the above polynomials, viz., by

$$H_x := \{\, t \in \mathbb{R}^n \, : \, p_x(t) = 0 \,\}\,, \quad x \in X.$$

We will assume that (λ_x) are chosen so that the intersection of any collection of $n+1$ hyperplanes is empty. Note that different choices of (λ_x) result possibly in hyperplane arrangements with different geometries. For example, in Example 1, we have the following 3 hyperplane arrangements with (at least) two different geometries.

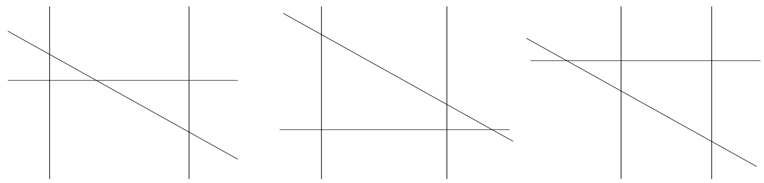


Figure 3: Hyperplane arrangements

We will focus on three geometrical objects in the hyperplane arrangement.

1. $V(X)$: the set of vertices

2. $CC(X)$: the set of n-dimensional connected components

3. $BCC(X)$: the set of n-dimensional bounded connected components

We claim that the following relations hold:

$$\#V(X) = \#\mathbb{B}(X), \quad \#CC(X) = \#\mathbb{I}(X), \quad \#BCC(X) = \#\mathbb{B}_-(X).$$

The first relation is straightforward. The other two are less so, but we will not prove them (and will not use them).

Next, let $\mathbb{I}_k(X)$ be the collection of independent sets of X whose cardinality is k. Then

$$\mathbb{I}(X) = \dot{\cup}_{k=0}^n \mathbb{I}_k(X).$$

Now, define a polynomial:

$$P_X(t) := \sum_{k=0}^{n} (\#\mathbb{I}_k(X))\, t^{n-k}, \quad t \in \mathbb{R}.$$

Then $P_X(1) = \#\mathbb{I}(X), \quad P_X(-1) = \#\mathbb{B}_-(X)$. The first statement here is trivial. The other one is a way to prove the $\#\mathbb{B}_-(X)$ is independent of the order on X.

3 Zonotopes: the dual geometry of hyperplane arrangements.

Now, let's consider X as a map:

$$X : \mathbb{R}^X \to \mathbb{R}^n : t \mapsto \sum_{x \in X} t_x x.$$

Then the zonotope of X is defined by

$$Z(X) := \mathrm{Im}(X|_{[0,1]^X}).$$

Assuming X to be unimodular, we have that

1. $\mathrm{vol}(Z(X)) = \#\mathbb{B}(X)$,

2. $\#(Z(X) \cap \mathbb{Z}^n) = \#\mathbb{I}(X)$,

3. $\#(\mathrm{int}(Z(X)) \cap \mathbb{Z}^n) = \#\mathbb{B}_-(X)$.

Example 3.1 To find the zonotope of X in Example 1, set x_i to be the ith column vector of X:

$$X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} =: [x_1\ x_2\ x_3\ x_4].$$

Start at origin and move in the direction of x_1, x_2, x_4, and x_3. And then go back in reverse.

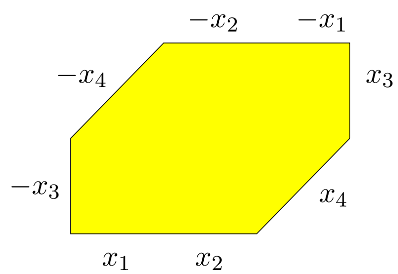


Figure 4: Zonotope

Every open zonotope $Z(X)$ is the disjoint union (up to a nullset) of the translated parallelepipeds

$$t_B + Z(B), \quad B \in \mathbb{B}(X).$$

There are multiple ways to choose the translations (t_B), and each such complete choice yield a tiling of the zonotope. Three such tilings of the above zonotope are displayed below. Each tiling corresponds to an ordering of X, and each such ordering corresponds to a different geometry on the hyperplane arrangement. In this duality, the vertices of the hyperplane arrangements are associated with the parallelepipeds that tile the zonotope, the bounded regions of the arrangement correspond to the interior lattice points in the zonotope, and the unbounded regions of the arrangement correspond to the lattice points on the boundary of the zonotope. Thus, for example, the number of vertices that belong to a connected region of the arrangement, must agree with the number of parallelepipeds that contain a given lattice point of the zonotope in their closure.

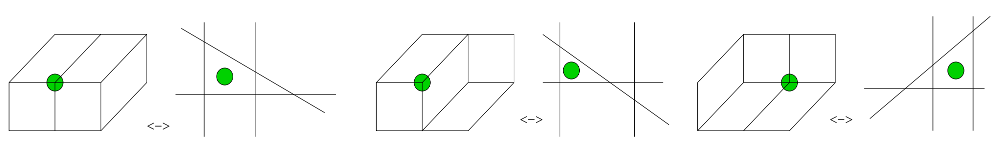


Figure 5: Tiling and hyperplane arrangements

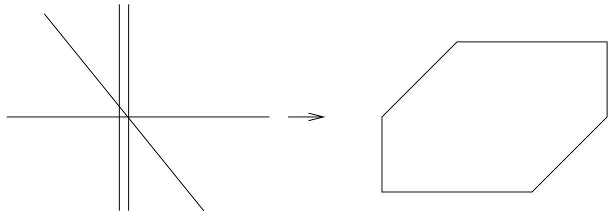


Figure 6: The central arrangement corresponds to no tiling

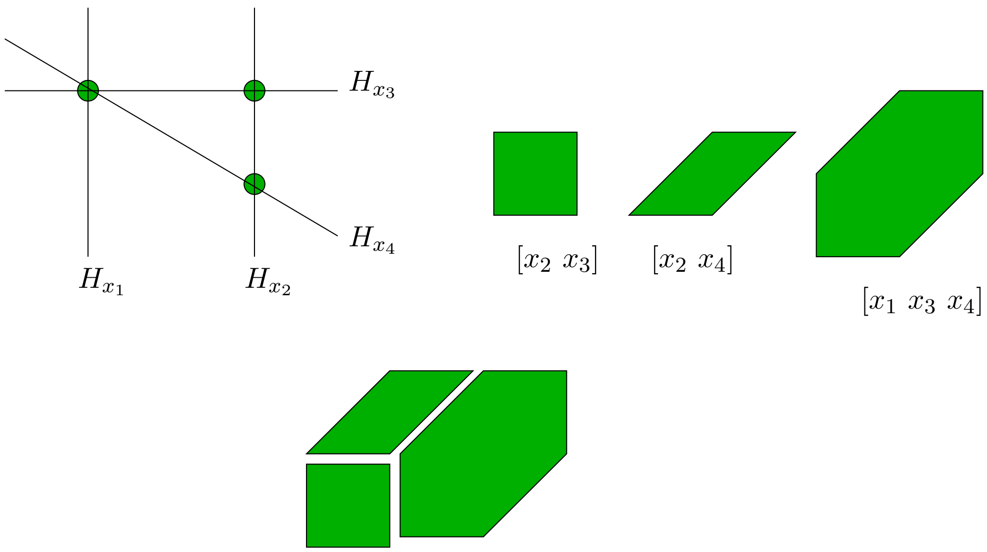


Figure 7: Tiling with 2 bases and 1 smaller zonotope