Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

Lectures given by Amos Ron
and recorded by Yeon Hyang Kim

Lecture 2: February 2, 2007

1 Notations.

Let $\sigma \subset \mathbb{R}^n$ be a finite subset (not a multiset). We are looking for a polynomial subspace

$$\Pi(\sigma) \subset C[t_1, \cdots t_n] =: \Pi$$

such that the restriction map $\Pi(\sigma) \to \mathbb{C}^\sigma \, : \, p \mapsto p|\sigma$ is an isomorphism. In particular, we must have $\dim \Pi(\sigma) = \#\sigma$.

To this end, we look at the point evaluation functional

$$\delta_\alpha : \Pi \to \mathbb{C} \, : \, p \mapsto p(\alpha),$$

and define

$$\Lambda := \operatorname{span} \{ \delta_\alpha \, : \, \alpha \in \sigma \}.$$

Then Λ is a subspace of the dual space Π' of Π.

2 Goal.

Our goal can now be restated as follows: For given σ, we want to find $F \subset \Pi$ such that

$$\Lambda|_F \cong F',$$

where F' is the dual space of F. We approach this problem by associating each $p \in \Pi$ with a differential operator:

$$p(D) := p\left(\frac{\partial}{\partial t_1}, \cdots, \frac{\partial}{\partial t_n}\right).$$

If $p(t) = x \cdot t, \quad x \in \mathrm{I\!R}^n \backslash \{ 0 \}$, then $p(D)$ is the directional derivative in the x-direction.
Then, for an entire function f, we define a pairing:

$$\langle p, f \rangle := p(D)f(0).$$

The representation of δ_α in this pairing is done by the exponential $e_\alpha : \mathrm{I\!R}^n \to \mathrm{I\!R} \, : \, t \mapsto e^{\alpha \cdot t}$; i.e.,

$$\langle p, e_\alpha \rangle = p(\alpha) = \delta_\alpha p.$$

Thus the abstract space Λ is represented here concretely by the exponential space

$$\mathrm{Exp}(\sigma) := \mathrm{span} \, \{ \, e_\alpha \, : \, \alpha \in \sigma \, \} \, .$$

This allows us to look at the power expansion of the exponentials. (this is not a big deal: the coefficients in the power expansion are essentially the action of δ_α on the monomial basis for Π. The only "real achievement" here that we get for free the correct normalization of the monomials that works for us). For each $f \in \mathrm{Exp}(\sigma)$, we write the power series expansion of f in the form

$$f = f_0 + f_1 + f_2 + \cdots,$$

where f_j is a homogeneous polynomial of degree j.
We then define a non-linear action $f \mapsto f_\downarrow$ by

$$f_\downarrow := f_j, \quad f_j \neq 0, \; f_i = 0, \; i < j.$$

I.e., $f_\downarrow$ is the first non-zero term in the above expansion of f $(0_\downarrow := 0)$. Finally, we define

$$\Pi(\sigma) := \mathrm{span} \, \{ \, f_\downarrow \, : \, f \in \mathrm{Exp}(\sigma) \, \} \, .$$

Example 2.1

$$(e_\alpha)_\downarrow = 1,$$
$$(e_\alpha - e_{\alpha'})_\downarrow : t \mapsto (\alpha - \alpha')t.$$

□

Question: What is the dimension of $\Pi(\sigma)$?

For an entire function f, let $T_j f$ be the j-th degree Taylor expansion of f at 0 (T_j is obviously linear):

$$T_j f := f_0 + f_1 + \cdots + f_j.$$

Note that $\deg(f_\downarrow) = j$ iff $f \in \ker T_{j-1} \backslash \ker T_j$. Thus, $\dim \operatorname{span}\{f_\downarrow : f \in \operatorname{Exp}(\sigma),\ \deg(f_\downarrow) = j\}$ is, with T'_j the restriction of T_j to $\ker T_{j-1}$,

$$\operatorname{rank} T'_j = \dim \ker T_{j-1} - \dim \ker T_j.$$

Summing from $j = 0$ to ∞ (where $T_{-1} := 0$), we easily obtain that

$$\sum_{j=0}^{\infty} \dim(\ker T_{j-1}) - \dim(\ker T_j) = \dim(\ker T_{-1}) = \dim \operatorname{Exp}(\sigma) = \#\sigma.$$

Here we used the fact that every finite set of exponentials is linearly independent.

Note that, for any analytic function $f \neq 0$,

$$\langle f_\downarrow, f \rangle \neq 0.$$

This means that there exista no $f \in \operatorname{Exp}(\sigma)\backslash\{0\}$ that satisfies

$$\langle p, f \rangle = 0, \quad \forall p \in \Pi(\sigma);$$

(choose $p := f_\downarrow \in \Pi(\sigma)$). Thus,

$$\dim \Pi(\sigma) = \dim \operatorname{Exp}(\sigma),$$

and no non-zero linear functional $f \in \operatorname{Exp}(\sigma)$ vanishes on $\Pi(\sigma)$. Thus we proved:

Theorem 2.2 *The map* $\operatorname{Exp}(\sigma) \to \Pi(\sigma)'$ *given by the above pairing is an isomorphism, hence* $\Pi(\sigma)$ *solves our original problem.*

Note that the pairing $\langle \cdot, \cdot \rangle$ allows us to identify any entire function f as a functional in Π'. Also, the action $f \mapsto f_\downarrow$ is defined for every entire function, hence the definition

$$F_\downarrow := \operatorname{span}\{\, f_\downarrow \,:\, f \in F \,\}$$

makes sense for any space of entire functions. This explains the meaning of the left-hand side space in our next theorem (albeit, a closer scrutiny reveals that the left-hand-side space is still an exponential space).

Now, given any non-zero $p \in \Pi$, let $p_\uparrow$ to be the highest degree homogeneous polynomial in p, i.e., $p_\uparrow$ is homogeneous, and $\deg(p - p_\uparrow) < \deg p$. Then we state without proof the following theorem:

Theorem 2.3 *Let* $p \in \Pi$ *be fixed. Then*

$$(p(D)\,(\operatorname{Exp}(\sigma)))_\downarrow \supset p_\uparrow(D)(\Pi(\sigma)).$$

□

The theorem can be used as follows: suppose that we hold in hand a homogeneous polynomial p, and we would like to understand the action of $p(D)$ on $\Pi(\sigma)$. The way to go is to try to find a non-homogeneous polynomial q such that (i) $q_\uparrow = p$, and (ii) q vanishes at as many points as possible of σ. Indeed, one easily verifies that

$$q(D)\mathrm{Exp}(\sigma) = \mathrm{Exp}(\sigma\backslash Z_q),$$

with Z_q the zero-set of q. Thus, the above theorem, in particular, tells us that

$$\dim(p(D)\Pi(\sigma)) \leq \#(\sigma\backslash Z_q),$$

with q any *lower order perturbation* of p, as above.

As a special case, we get the following important corollary:

Theorem 2.4 *If p vanishes on σ then $p_\uparrow$ annihilates $\Pi(\sigma)$.* $\square$

Setup: The theorem gives us a handy tool for estimating below dimensions of spaces of the form

$$D = \{f \in \Pi : p(D)f = 0, \ \forall p \in I\},$$

with I some finite set of homogeneous polynomials. (We refer to spaces as the above D as (joint) kernels of differential operators). The idea is as follows, for each $p \in I$, we define (very carefully!!) a polynomial q_p such that $q_{p\uparrow} = p$. Then we find the common zero set σ of the polynomials $(q_p : p \in I)$. Since each q_p vanishes on σ, then, by the above theorem, each $p(D)$ annihilates $\Pi(\sigma)$, hence

$$\Pi(\sigma) \subset D.$$

Since $\dim \Pi(\sigma) = \#\sigma$, we obtain in this way the lower bound

$$\dim D \geq \#\sigma.$$

We conclude with a concrete example (actually, two).

Example 2.5

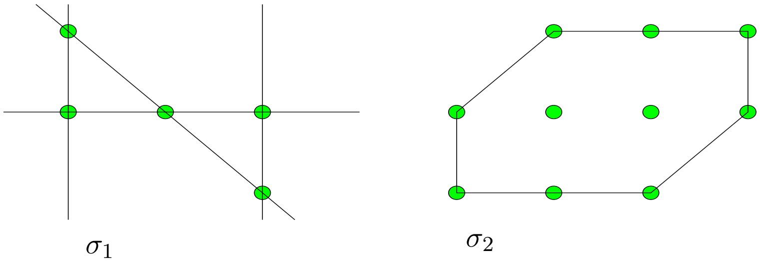

Figure 8: $\Pi(\sigma)$

$$\mathrm{Exp}(\sigma_1) = \mathrm{span}\,\{\,1, e^{t_1}, e^{t_2}, e^{2t_1-t_2}, e^{2t_1}\,\}$$

$$\Pi(\sigma_1) = \mathrm{span}\,\{\,1,\ t_1,\ t_2,\ t_1^2,\ t_2^2 - 2t_1t_2\,\}$$

$$\mathrm{Exp}(\sigma_2) = \mathrm{span}\,\{\,1, e^{t_1}, e^{t_2}, e^{2t_1}, e^{t_1+t_2}, e^{2t_1+t_2}, e^{3t_1+t_2}, e^{t_1+2t_2}, e^{2t_1+2t_2}, e^{3t_1+2t_2}\,\}$$

$$\Pi(\sigma_2) = \mathrm{span}\,\{\,1,\ t_1,\ t_2,\ t_1^2,\ t_1t_2,\ t_2^2,\ t_1^2t_2,\ t_1t_2^2,\ t_1^3,\ t_1^2t_2(t_1+t_2)\,\}$$

3 Notes on this lecture

This lecture, as a whole, is taken from the following two papers:

C. de Boor, A. Ron, On multivariate polynomial interpolation, Constructive Approximation **6**(1990), 287–302.

C. de Boor, A. Ron, On ideals of finite codimension and applications to box splines theory, Journal of Mathematical Analysis and its Applications **158** (1991), 168–193.

□