Lecture notes on:
Ideals over Hyperplane arrangements and Zonotopes.

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1 Our basic tool.

We stated last time the following result:

Corollary 1.1 *Let* $\sigma \subset \mathbb{R}^n$ *be a finite subset. Let* $p \in \Pi$. *If* $p|_\sigma = 0$, *then* $p_\uparrow(D)(\Pi(\sigma)) = 0$.

Here is the (strikingly simple) proof for this result:

Proof: Set $p =: p_\uparrow + p'$, then $\deg p' < \deg p$. For $f \in \mathrm{Exp}(\sigma)$, set $f =: f_\downarrow + f'$. Then we have

$$p(D)(\mathrm{Exp}(\sigma)) = 0.$$

Consequently,

$$p(D)f = 0 = p_\uparrow(D)f_\downarrow + \text{ h.o.t.}$$

(Here "h.o.t" stands for "higher order terms"). Therefore,

$$p_\uparrow(D)f_\downarrow = 0.$$

□

2 The space $D_{\mathbb{B}'}(X)$.

Recall that, for a finite multiset $X \subset \mathbb{R}^n \backslash \{0\}$ of full rank n, we have

$$\#V(X) = \#\mathbb{B}(X),$$

where, for each $x \in X$, x is associated with a non-homogeneous polynomial $p_x : \mathbb{R}^n \to \mathbb{R} : t \mapsto x \cdot t - \lambda_x$, and $V(X)$ is the vertex set of the corresponding hyperplane arrangement. For a multi-subset $Y \subset X$, we define

$$p_Y := \prod_{y \in Y} p_y,$$

$$q_Y := \prod_{y \in Y} q_y,$$

where $q_y : \mathbb{R}^n \to \mathbb{R} : t \mapsto y \cdot t$. Also, we define

$$q_Y(D) := \prod_{y \in Y} D_y,$$

where D_y is the directional derivative in the y-direction.

Note, that there is a natural bijection $B \mapsto v_B$ between $\mathbb{B}(X)$ and $V(X)$, where each $B \in \mathbb{B}(X)$ is mapped to the unique common zero v_B of $(p_y : \ y \in B)$. (The injectivity of the map is valid only for a generic selection of (λ_y); we deal herein only with that generic case). This implies that each subset $\mathbb{B}'$ of $\mathbb{B}(X)$ is associated in unique way with $V' \subset V(X)$. We define

$$\mathcal{D}_{\mathbb{B}'}(X) := \left\{ f \in \Pi \, : \, q_Y(D)f = 0, \quad \forall Y \subset X \text{ such that } Y \cap B \neq \emptyset, \ \forall B \in \mathbb{B}' \right\}.$$

Then we have the following theorem:

Theorem 2.1 *Let $X \subset \mathbb{R}^n \backslash \{0\}$ be a finite multiset of full rank n. Then, for any $\mathbb{B}'$ of $\mathbb{B}(X)$,*

$$\dim \mathcal{D}_{\mathbb{B}'}(X) \geq \#\mathbb{B}'.$$

Proof: Note that for $Y \subset X$ and $B \in \mathbb{B}(X)$, $p_Y(v_B) = 0$ iff $Y \cap B \neq \emptyset$. Now, set $V' := \{v_B : \ B \in \mathbb{B}'\}$, and let Yq be a multi-subset of X such that

$$Y \cap B \neq \emptyset, \ \forall B \in \mathbb{B}'.$$

Then we conclude that p_Y vanishes on V'. Then, by Theorem 1.1, we have

$$((p_Y)_\uparrow)(D)(\Pi(V')) = 0.$$

Since $(p_Y)_\uparrow = q_Y$, we conclude that

$$\Pi(V') \subset \mathcal{D}_{\mathbb{B}'}(X).$$

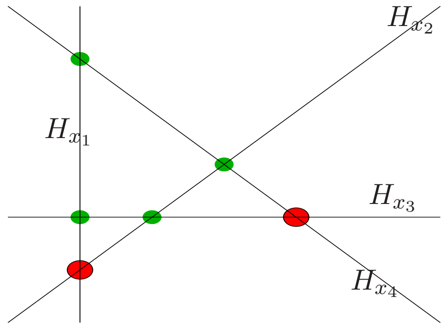
But,

$$\dim \Pi(V') = \#V' = \#\mathbb{B}'.$$

□

Example 2.2 Let

$$X := \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \end{bmatrix} =: [x_1, x_2, x_3, x_4], \quad \mathbb{B}' = \{ [x_1 \ x_2], \ [x_3 \ x_4] \},$$



$$p_{x_1}(t) := t_1$$
$$p_{x_1}(t) := t_1 - t_2 - 1$$
$$p_{x_1}(t) := t_2$$
$$p_{x_1}(t) := t_1 + t_2 - 3$$

Figure 9: Hyperplanes of X

Then, for $Y \subset X$ to intersect both bases in $\mathbb{B}'$, we clearly must have $\#Y \geq 2$. Thus, $q_Y(D)$ annihilates Π_1, hence

$$\dim D_{\mathbb{B}'}(X) \geq 3 > \#\mathbb{B}'.$$

□

3 $\mathcal{D}(X)$ and $\mathcal{P}(X)$.

We have seen the geometric duality between the hyperplane arrangement and the zonotope that are associated with the multiset X. We will now develop an algebraic counterpart of that duality. The algebraic structures will come in three pairs of polynomials spaces. Each pair will be shown to be dual space of its pairmate via our pairing $\langle \cdot, \cdot \rangle$.

The first pair will be referred to as the central pair of X. The space $\mathcal{P}(X)$ below is the central space of the zonotope $Z(X)$, while the space $\mathcal{D}(X)$ is the central space of the hyperplane arrangement.

In order to define those two spaces, we decompose the power set 2^X into the following two disjoint subsets: the set of long subsets ($L(X)$) and the set of short subsets ($S(X)$):

$$L(X) := \{ Y \subset X \,:\, Y \cap B \neq \emptyset, \quad \forall B \in \mathbb{B}(X) \}.$$

$$S(X) := \{ Y \subset X \,:\, \mathrm{rank}(X \backslash Y) = n \}.$$

Define

$$\mathcal{D}(X) := \{ p \in \Pi \,:\, q_Y(D)p = 0, \quad \forall Y \in L(X) \},$$

$$\mathcal{P}(X) := \mathrm{span}\{ q_Y \,:\, Y \in S(X) \}.$$

Then we have the following theorem:

Theorem 3.1 *(1)* $\dim \mathcal{D}(X) = \dim \mathcal{P}(X) = \#\mathbb{B}(X)$, *(2) The map*

$$\mathcal{D}(X) \to \mathcal{P}(X)' : p \mapsto \langle p, \cdot \rangle$$

is an isomorphism.

4 Combinatorics: the central Hilbert function of X

Let Π_j^0 be the space of homogeneous polynomials of degree j (in n variables). The asserted isomorphism (part (2)) in the theorem above implies (since both $\mathcal{P}(X)$ and $\mathcal{D}(X)$ are *homogeneous*, i.e., are spanned by homogeneous polynomials), that, for every j

$$\dim(\Pi_j^0 \cap \mathcal{D}(X)) = \dim(\Pi_j^0 \cap \mathcal{P}(X))$$

We refer to the homogeneous dimensions of the space $\mathcal{P}(X)$ as the *central Hilbert function of* X, i.e.,

$$h_X : \mathbb{Z}_+ \to \mathbb{Z}_+ \ : \ j \mapsto \dim(\Pi_j^0 \cap \mathcal{P}(X)).$$

Note that

$$\sum_j h_X(j) = \#\mathbb{B}(X).$$

We will show later that h_X can be computed directly by studying the dependence/independence relations among the vectors in X.

Example 4.1 *Let*

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}.$$

Then $\#\mathbb{B}(X) = 16$*, and*

$$h_X = (1, 3, 6, 6).$$

□

5 Notes on Lectures 2 and 3

Lecture 2, as a whole, is taken from the following two papers:

C. de Boor, A. Ron, On multivariate polynomial interpolation, Constructive Approximation **6***(1990), 287–302.*

C. de Boor, A. Ron, On ideals of finite codimension and applications to box splines theory, Journal of Mathematical Analysis and its Applications **158** *(1991), 168–193.*

Theorem 2.1 in the present lecture, as well as the given proof are taken from the latter paper.

Obviously, the theorem implies as a special case that

$$\dim \mathcal{D}(X) \geq \#\mathbb{B}(X).$$

This specific inequality was first proved by Dahmen and Micchelli (Studia Math., 1985) by induction on $\#X$ and on n. A non-inductive analytic-type argument appears in Ben Artzi and Ron (TAMS, 1988).

The assertion in Theorem 3.1 that $\dim \mathcal{D}(X) = \#B(X)$ is due to Dahmen and Micchelli (Studia Math., 1985). They subsequently provided (in Adv. Math., 1989) and very elegant proof for the inequality $\dim \mathcal{D}(X) \leq \#\mathbb{B}(X)$, that uses the matroidal structure of X.

The space $\mathcal{P}(X)$ was introduced independently by Hakopian (1989), and by Dyn and Ron (TAMS, 1990). The central Hilbert function is introduced in the last section of the latter paper.