CS726 - Nonlinear Optimization I - Homework I

September 12, 2012

This assignment is due at the beginning of class on September 19.

1. Let $A = [A_{ij}]$ and $B = [B_{ij}]$ be $n \times n$ positive semidefinite matrices. Define the *Hadamard Product* between A and B as $A \circ B = [A_{ij}B_{ij}]$. Prove that $A \circ B$ is positive semidefinite.

2. Let p be a probability distribution on the interval $[0, 1]$. Let the kth *moment* of p be the expected value

$$\mu_k = \mathbb{E}[x^k] = \int_0^1 x^k p(x)dx \,.$$

Prove that the matrix

$$\begin{bmatrix} \mu_0 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{bmatrix}$$

is positive semidefinite.

3. Let $C \subset \mathbb{R}^n$ and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

(a) For $a \geq 0$ we define C_a as $\{x \; : \; \mathrm{dist}(x, C) \leq a\}$, where $\mathrm{dist}(x, C) = \inf_{z \in C} \|x - z\|$. That is, C_a is the set of all points with distance at most a from C. Show that if C is convex, then C_a is convex.

(b) For $a \geq 0$ we define $C_{-a} = \{x \; : \; B(x; a) \subset C\}$, where $B(x; a)$ is the ball (in the norm $\|\cdot\|$), centered at x, with radius a. C_{-a} consists of all points that are at least a distance a from the complement of C, $\mathbb{R}^n \setminus C$. Show that if C is convex, then C_{-a} is convex.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is convex and differentiable. Show that its running average F, defined as

$$F(x) = \frac{1}{x} \int_0^x f(t)dt$$

is convex on $\{x \; : \; x > 0\}$.