CS726 - Nonlinear Optimization I - Homework 6

October 24, 2012

This assignment is due at the beginning of class on October 31 (boo!).

1. Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x_1, x_2) \\ \text{subject to} & 2x_1 + x_2 \geq 1 \\ & x_1 + 3x_2 \geq 1 \\ & x_1 \geq 0, x_2 \geq 0 \end{array}$$

Make a sketch of the feasible set. For each of the following objective functions, give the optimal set and the optimal value.

(a) $f_0(x_1, x_2) = x_1 + x_2$

(b) $f_0(x_1, x_2) = -x_1 - x_2$

(c) $f_0(x_1, x_2) = x_1$

(d) $f_0(x_1, x_2) = \max\{x_1, x_2\}$

(e) $f_0(x_1, x_2) = x_1^2 + 9x_2^2$

2. Consider the linear program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \leq b \end{array}$$

where A is nonsingular and square. Show that the optimal value is given by

$$p^\star = \begin{cases} c^T A^{-1} b & \text{if } (A^{-1})^T c \leq 0 \\ -\infty & \text{otherwise} \end{cases}.$$

3. For a norm $\|\cdot\|$, the dual norm is the function

$$\|z\|_* = \sup\{x^T z \ : \ \|x\| \leq 1\}$$

(a) Prove that $\|\cdot\|_*$ is a norm if $\|\cdot\|$ is a norm.

(b) Prove that the dual norm of the ℓ_2 norm is the ℓ_2 norm by computing the dual problem of

$$\begin{array}{ll} \text{minimize} & -z^T x \\ \text{subject to} & \sum_{i=1}^n x_i^2 \leq 1 \end{array}.$$

(c) Prove that the dual norm of the ℓ_1 norm is the ℓ_∞ norm by first showing that $\|x\|_1 \leq s$ if and only if there exists a vector t satisfying

$$\begin{array}{ll} -t_i \leq x_i \leq t_i & i = 1, \ldots, n \\ \sum_{i=1}^n t_i \leq s & \end{array} ,$$

and then computing the dual problem of

$$\begin{array}{ll} \text{minimize}_{(x,t)} & -z^T x \\ \text{subject to} & -t_i \leq x_i \leq t_i \ \text{ for } \ i = 1, \ldots, n \\ & \sum_{i=1}^n t_i \leq 1 \end{array} .$$

4. Derive the dual problem of

$$\begin{array}{lll} \text{minimize}_{(x,y)} & -\sum_{i=1}^m \log(y_i) & \\ \text{subject to} & a_i^T x - b_i + y_i = 0 & i = 1, \ldots, m \\ & y_i \geq 0 & i = 1, \ldots, m\,. \end{array}$$