Exploiting symmetry for the distributed control of spatially interconnected systems

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This paper discusses how distributed controller design and analysis for spatially interconnected systems can be enhanced by considering the symmetry of the interconnection system. Introducing techniques from group theory, we show how to extend the LMI results of D'Andrea to systems with a high level of symmetry. We also show how to exploit the form of the symmetry group to produce a hierarchy of decreasingly conservative tests.

I. INTRODUCTION

II. NOTATION, DEFINITIONS, AND PRELIMINARIES

A detailed and careful introduction to abstract groups can be found in [6], but we will need only a few definitions. A *group* is a set with a composition rule with identity such that every element is invertible. A *subgroup* of $\mathbb{G}$ is a subset $H \subset \mathbb{G}$ such that H is a group under the composition operator of $\mathbb{G}$. A subgroup N is *normal* in $\mathbb{G}$ if the set $gNg^{-1} = \{gng^{-1} | n \in N\} = N$. The quotient group of $\mathbb{G}$ by a normal subgroup N is denoted $\mathbb{G}/N$ and can be produced by imposing the relations $n = 1$ on $\mathbb{G}$.

A subset, S, of a group, $\mathbb{G}$,is a set of generators for $\mathbb{G}$ if every element $g \in \mathbb{G}$ can be written as a product of powers of elements from S. A relation, r, is a product of elements of $\mathbb{G}$ which equals the identity. A group is *finitely presented* if there exist a set of generators S and set of relations amongst the generators R such that any relation r' for $\mathbb{G}$ can be written as a product of relations in R or their inverses. A group presented by generators S and relations R is denoted $\mathbb{G} = \langle S|R \rangle$.

The *Cayley Graph*, G, of a finitely presented group $\mathbb{G} = \langle S|R \rangle$ is a directed graph with the set of elements of $\mathbb{G}$ as vertices and with a directed edge from g to h if there is an $s \in S$ such that $h = gs$. Note that this definition means that two different presentations of $\mathbb{G}$ by generators and relations give rise to different Cayley graphs.

We will be dealing with signals moving through the Cayley Graph and external disturbances. That is, vector valued functions on $\mathbb{R} \times \mathbb{G}$, $u(t, g)$. Here t denotes the temporal variable and g denotes a vertex in the Cayley Graph. Formally, we define l_2 to be the Hilbert space of all functions from $x : \mathbb{G} \to \mathbb{R}^\bullet$ such that the quantity

$$\sum_{g \in \mathbb{G}} x(g)^* x(g) \tag{1}$$

is finite. The Hilbert space $\mathcal{L}_2$ will denote the space of functions $u : \mathbb{R}^+ \to l_2$ such that

$$\int_0^\infty \|u(t)\|_{l_2} dt \tag{2}$$

is finite.

We can write any $u \in \mathcal{L}_2$ as a function on $\mathbb{R} \times G$, $u = u(t, g)$. Under this labeling, we can view u as having a temporal and spatial component with time indicated by the real variable t and g denoting a spatial location on the Cayley graph. The group $\mathbb{G}$ then acts naturally as a set of unitary spatial-shift operators on these signals

$$(\hat{\mathbf{g}} u)(t, h) = u(t, hg) \tag{3}$$

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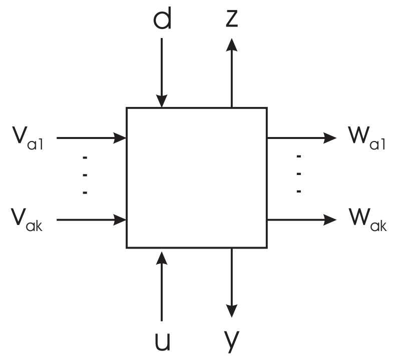

FIG. 1: A basic building block

III. INTERCONNECTED SYSTEMS ON CAYLEY GRAPHS

The systems considered herein consist of collections of identical linear systems connected by the structure of a Cayley graph. Let $\mathbb{G} = \langle a_1, \ldots, a_K | R(a_1, \ldots, a_K)\rangle$ be a finitely presented group. Our basic building blocks will be finite dimensional, linear, time-invariant systems on the Cayley graph of $\mathbb{G}$ with the form

$$\begin{bmatrix} \dot{x}(t,g) \\ w(t,g) \\ z(t,g) \\ y(t,g) \end{bmatrix} = \begin{bmatrix} A_{TT} & A_{ST} & B_T^d & B_T^u \\ A_{TS} & A_{SS} & B_S^d & B_S^u \\ C_T^z & C_S^z & D_S^{zd} & D_S^{zu} \\ C_T^y & C_S^y & D_S^{yd} & D_S^{yu} \end{bmatrix} \begin{bmatrix} x(t,g) \\ v(t,g) \\ d(t,g) \\ u(t,g) \end{bmatrix} \tag{4}$$

for a fixed g. For any building block the vectors $v(t,g)$ and $w(t,g)$ can be partitioned as

$$v(t,g) = \begin{bmatrix} v_{a_1}(t,g) \\ v_{-a_1}(t,g) \\ \vdots \\ v_{a_K}(t,g) \\ v_{-a_K}(t,g) \end{bmatrix}, \quad w(t,g) = \begin{bmatrix} w_{a_1}(t,g) \\ w_{-a_1}(t,g) \\ \vdots \\ w_{a_K}(t,g) \\ w_{-a_K}(t,g) \end{bmatrix} \tag{5}$$

This partitioning allows us to characterize the interconnection structure on G. We impose

$$w_{a_k}(t,g) = v_{a_k}(t, ga_k) \qquad w_{-a_k}(t,g) = v_{-a_k}(t, ga_k^{-1}) \tag{6}$$

The signals d and z denote external input and output, while the signals y and u will be utilized for closing the loop with a distributed controller in section V. The block diagram of such a system is shown in figure 1. For ease of writing later, the matrix in equation 5 will be denoted $\mathcal{M}_p$.

Note that the structure of the basic building block only depends on the number of generators of our abstract group $\mathbb{G}$, and a variety of connections can be created using the same building block with different group relations.We will discuss a few simple examples of interconnections in the next section.

Let m_k denote the size of the interconnection variables v_{a_k}. Then we can define the structured operator

$$\Delta \equiv \text{diag}\left(\hat{\mathbf{a}}_\mathbf{1} \mathbb{1}_{m_1}, \hat{\mathbf{a}}_\mathbf{1}^{-1} \mathbb{1}_{m_1}, \ldots, \hat{\mathbf{a}}_\mathbf{K} \mathbb{1}_{m_K} \hat{\mathbf{a}}_\mathbf{K}^{-1} \mathbb{1}_{m_K}\right) \tag{7}$$

and write the state-space equation for the interconnected system as

$$\begin{bmatrix} \dot{x}(t,g) \\ (\Delta v)(t,g) \\ z(t,g) \\ y(t,g) \end{bmatrix} = \begin{bmatrix} A_{TT} & A_{ST} & B_T^d & B_T^u \\ A_{TS} & A_{SS} & B_S^d & B_S^u \\ C_T^z & C_S^z & D_S^{zd} & D_S^{zu} \\ C_T^y & C_S^y & D_S^{yd} & D_S^{yu} \end{bmatrix} \begin{bmatrix} x(t,g) \\ v(t,g) \\ d(t,g) \\ u(t,g) \end{bmatrix} \tag{8}$$

and unless stated we assume $x_0 = 0$.

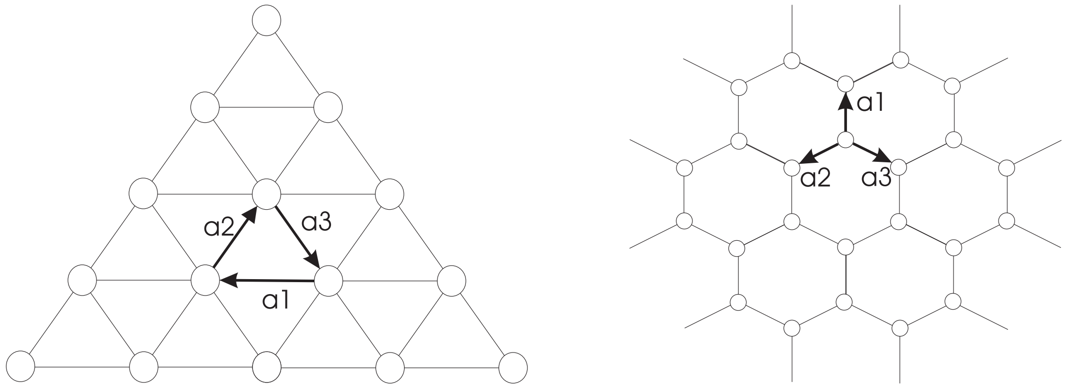

FIG. 2: Two possible planar interconnections of nodes with three generators

IV. EXAMPLES

The simplest interesting example is the infinite chain of identical units in one dimension. Here the group is $\mathbf{Z}$, the integers, and the generator is the element 1. That is, we can write the signals as $u(t,n)$ and the generator is a spatial right-shift $(\hat{\mathbf{a}}_{\mathbf{1}}u)(t,n) = u(t,n+1)$. We can immediately extend this result to a lattice in d dimensions by introducing d generators $a_1,\ldots,a_d$ with the relations $a_j a_k = a_k a_j$ for all k,j. Signals can be labeled as functions of $d+1$ variables $u(t,s_1,\ldots,s_d)$ and the generators are spatial shifts

$$(\hat{\mathbf{a}}_{\mathbf{j}}u)(t,s_1,\ldots,s_d) = u(t,s_1,\ldots,s_j+1,\ldots,s_d) \tag{9}$$

As we will see shortly, most of the work on distributed systems to date assumes a spatial structure of a lattice. To get a periodic structure, we may impose relations $a_k^{N_k} = 1$.

We can continue to produce examples. Consider groups with three generators a_1,a_2,a_3. The group presented by the relations

$$a_1a_2a_3 = a_3a_2a_1 = 1 \tag{10}$$

yields a triangular mesh in the plane. If instead we have

$$a_1^2 = a_2^2 = a_3^2 = (a_1a_2a_3)^2 = 1 \tag{11}$$

then we get a hexagonal mesh. Equation 10 generates the 2 dimensional space group **p1** while equation 11 generates **p2**. There are 17 such space groups in 2D, and 320 in 3D, and by introducing group theoretic abstractions we have expanded the class of interconnection structures for distributed systems [8], [9].

V. LINEAR MATRIX INEQUALITIES FOR CONTROLLER SYNTHESIS

For control design, we want to design a controller with the same structure as the plant. Our controllers have the form

$$\begin{bmatrix} \dot{x}(t,g) \\ (\Delta v)(t,g) \\ y(t,g) \end{bmatrix} = \begin{bmatrix} A^K_{TT} & A^K_{ST} & B^K_T \\ A^K_{TS} & A^K_{SS} & B^K_S \\ C^K_T & C^K_S & D^K \end{bmatrix} \begin{bmatrix} x(t,g) \\ v(t,g) \\ u(t,g) \end{bmatrix} \tag{12}$$

denoted by $\mathcal{M}_k$. When we connect the signals u and y of the controller and the plant, we call the closed loop system $\mathcal{M}$.

There are three properties that we desire the closed loop system to have.

- *Well-posedness*: Well-posedness describes the realizability of a system. A system is well posed, if when $x(0) = 0$ and $d = 0$ then the interconnection signals v and w are also identically zero.

- *Stability*: A system is stable if when $d = 0$, the norm of the signal x is bounded above by a decaying exponential $\alpha\exp(-\beta t)$.

- *Contractiveness*: A system is contractive if for any input signal $d \neq 0$, $\|d\| > \|z\|$.

These three combined yield our synthesis problem: given a distributed system $\mathcal{M}_p$, design a controller $\mathcal{M}_K$ such that the closed loop system $\mathcal{M}$ is well-posed, stable, and contractive.

The case where there are no generators reduces to the classic H_∞ control problem [5]. For the case of abelian groups (that is, groups where all of the generators commute), the analysis problem has been studied in detail in [3], [1], and [2]. In particular, the authors of [7] showed that a controller M_k existed if one could find a solution to the linear matrix inequality

$$\sum_{k=1}^{N} q_k A_k + A_0 < 0 \tag{13}$$

with decision variables q_k and data A_k, A_0 only a function of the open loop plant data $\mathcal{M}_p$. Such a linear matrix inequality can be solved efficiently using semidefinite programming [4]. Furthermore, when a solution exists, they show how to construct $\mathcal{M}_K$ from the solutions q_k.

In the proof of the sufficiency of this LMI test, the authors of [7] only use the fact that the Δ operator is unitary. The commutative relations of the lattice group never come into play. Indeed, any controller synthesis can be performed using the LMI 13 for any system described by 8 with a unitary operator Δ which has the block diagonal structure

$$\Delta = \left(\Delta_1 \mathbb{1}_{m_1}, \ldots, \Delta_{-K} \mathbb{1}_{m_{-K}}\right) \tag{14}$$

By defining our system in this more general fashion, we extend the possibility of studying controller design on a much larger class of distributed systems. As an easy example, note that the group **p2** is not commutative.

VI. USING RELATIONS TO GENERATE LESS CONSERVATIVE ANALYSES

The results proven above by looking only at the basic building block are inherently conservative and never examine the structure of the interconnections of the system. We now will explore how to create LMI tests which are less conservative by grouping building blocks into subsystems.

Let N be a normal subgroup of finite index generated by adding new relations to the relation set of $\mathbb{G}$ and let $\eta : \mathbb{G} \to \mathbb{G}/N$ be the quotient map. Then the quotient space $\mathbb{G}/N$ inherits a finite group structure with a Cayley graph generated by the generators of $\mathbb{G}$. For any $p \in \mathbb{G}/N$, and $n \in N$, there exists a unique $\lambda(n,p) \in \mathbb{G}$ such that $\lambda(n,p) = gn$ and $\eta(gn) = p$. λ is called the *lifting* map. Also, for $n \in N \leq \mathbb{G}$, there is a unique lifting of the Cayley graph of $\mathbb{G}/N$ to a subgraph of the Cayley graph of $\mathbb{G}$ called a *transversal* [10]. These subgraphs generate $\mathbb{G}$ using translations only contained in N. That is, if R is the transversal lifted to the identity, then $NR = G$. Furthermore if $\hat{\mathbf{n}}R$ is the transversal of the identity translated by $n \in N$, then $\hat{\mathbf{n}}R$ is the transversal lifted from $\mathbb{G}/N$ to n and

$$G = \cup_{n \in N} \hat{\mathbf{n}} R \qquad \text{and} \qquad \hat{\mathbf{n}} R \cap \hat{\mathbf{m}} R = \qquad \forall n \neq m \in N \tag{15}$$

As an example of such a decomposition, consider again the group **p2** of section IV. Take the subgroup generated by the three elements

$$a_2 a_3 a_2 a_1, \quad a_3 a_1 a_3 a_2, \quad \text{and} \quad a_1 a_2 a_1 a_3 \tag{16}$$

This subgroup is normal, has index 6, and is isomorphic to **p1**. The transversals are hexagons, and the quotient group is isomorphic to S_3 the permutation group on three letters. This grouping is illustrated in figure 3.

Take the set of edges in the transversal to be *interior* edges. The set of all edges connected to nodes in the transversal which are not interior edges are called *exterior* edges. We can shift between any two nodes in a transversal by first projecting into the graph of $\mathbb{G}/N$ and then lifting back to our transversal. Indeed, for each element in the group, we can define a new shift operator based on this operation

$$\Pi(\hat{\mathbf{h}}) u(t,g) = u(t, \lambda(\eta(gh), g) \tag{17}$$

We can group all of the signals involving edges in R as follows. Stack the set of internal state variables in a vector

$$x(t,n) = \left[x(t,n), x(t, \lambda(n, \rho_1)), \ldots x(t, \lambda(n, \rho_M))\right], \qquad \rho_k \in \mathbb{G}/N \tag{18}$$

and the disturbance and output signals can be stacked accordingly. Group the interior signals as v_I and the exterior signals as v_E. Under this identification, the system can be written with the state space formulation

$$\begin{bmatrix} \dot{x}(t,n) \\ (\Pi v_I)(t,n) \\ (\Theta v_E)(t,n) \\ z(t,g) \end{bmatrix} = \begin{bmatrix} A_{TT} & A_{TI} & A_{TE} & B_T \\ A_{IT} & A_{II} & A_{EI} & B_I \\ A_{ET} & A_{IE} & A_{EE} & B_E \\ C_T & C_I & C_E & D \end{bmatrix} \begin{bmatrix} x(t,n) \\ v_I(t,n) \\ v_E(t,n) \\ d(t,n) \end{bmatrix} \tag{19}$$

If we define the matrices

$$\begin{bmatrix} \tilde{A}_{TT} & \tilde{A}_{ST} & \tilde{B}_T \\ \tilde{A}_{TS} & \tilde{A}_{SS} & \tilde{B}_S \\ \tilde{C}_T & \tilde{C}_S & \tilde{D} \end{bmatrix} = \begin{bmatrix} A_{TT} & A_{TE} & B_T \\ A_{ET} & A_{EE} & B_E \\ C_T & C_E & D \end{bmatrix} + \begin{bmatrix} A_{TI} \\ A_{EI} \\ C_I \end{bmatrix} (\Pi - A_{II})^{-1} \begin{bmatrix} A_{IT} & A_{IE} & B_I \end{bmatrix} \tag{20}$$

then the variables v_I can be eliminated to yield the equivalent formulation

$$\begin{bmatrix} \dot{x}(t,n) \\ (\Theta v_E)(t,n) \\ d(t,n) \end{bmatrix} = \begin{bmatrix} \tilde{A}_{TT} & \tilde{A}_{ST} & \tilde{B}_T \\ \tilde{A}_{TS} & \tilde{A}_{SS} & \tilde{B}_S \\ \tilde{C}_T & \tilde{C}_S & \tilde{D} \end{bmatrix} \begin{bmatrix} x(t,n) \\ v_E(t,n) \\ z(t,n) \end{bmatrix} \tag{21}$$

This can new realization of our system can now be fed into the LMI tests of the previous section and new bounds on performance can be estimated.

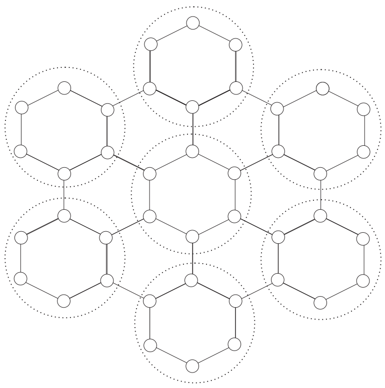

FIG. 3: Subdividing the hexagonal tesselation into subsystems for less conservative analysis

VII. CONCLUSIONS

Future work includes a careful analysis of how combining the same building block in different configurations affects performance and stability.

The grouping process may be repeated *ad infinitum.* Once the subsystems are grouped, normal subgroups of the normal subgroup can be used to repeat this analysis. For example, we grouped hexagons into a group with the structure of the hexagons of the **p1**. The triangles in this new grouping can be combined to find a group with the structure of **p2** again. Of course, the state space form of the matrix $\mathcal{M}_p$ will increase in size, so it determining the trade-off between conservatism and LMI complexity is an important consideration for analysis. Furthermore, these new talk to all subunits which are grouped together, and this requirement may not be desirable for certain applications.

Acknowledgments

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