Necessary and Sufficient Conditions for Success of the Nuclear Norm Heuristic for Rank Minimization

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Abstract— **Rank minimization–minimizing the rank of a matrix subject to constraints–is a challenging problem that arises in many control applications including controller design, realization theory and model reduction. The general formulation of rank minimization subject to convex constraints is NP-HARD, and for most practical problems there are no efficient exact algorithms. A popular heuristic algorithm is to replace the rank function with the sum of the singular values of the decision variable. In this paper, we provide a necessary and sufficient condition that quantifies when this heuristic successfully finds the minimum rank solution of a linear constraint set. We further show that most of the problems of interest in control can be formulated as rank minimization subject to such linear constraints. We additionally provide a probability distribution over instances of the affine rank minimization problem such that instances sampled from this distribution satisfy our necessary conditions for success with overwhelming probability provided the number of constraints is appropriately large. Finally we give empirical evidence that these probabilistic bounds are realistic in numerical simulations.**

I. INTRODUCTION

Optimization problems involving constraints on the rank of matrices are ubiquitous in control applications, arising in the context of low-order controller design [8], [15], minimal realization theory [10], and model reduction [3]. Rank minimization is also of interest to a broader optimization community in a variety of applications including inference with partial information [18] and embedding in Euclidean spaces [13]. In certain instances with special structure, the rank minimization problem can be solved via the singular value decomposition or can be reduced to the solution of a linear system [15], [16]. In general, however minimizing the rank of a matrix subject to convex constraints is NP-HARD. The best exact algorithms for this problem involve quantifier elimination and such solution methods require doubly exponential time in the dimensions of the matrix variables.

A popular heuristic for solving rank minimization problems in the controls community is the "trace heuristic" where one minimizes the trace of a positive semidefinite decision variable instead of the rank (see, e.g., [3], [15]). A generalization of this heuristic to non-symmetric matrices introduced by Fazel in [9] minimizes the *nuclear norm*, or the sum of the singular values of the matrix, over the constraint set. When the matrix variable is symmetric and positive semidefinite, this heuristic is equivalent to the trace heuristic, as the trace of a positive semidefinite matrix is equal to the sum of its singular values. The nuclear norm is a convex function and can be optimized efficiently via semidefinite programming. Both the trace heuristic and the nuclear norm generalization have been observed to produce very low-rank solutions in practice, but until very recently, no known conditions were available for when the heuristic succeeded.

The the first non-trivial sufficient conditions that guaranteed the success of the nuclear norm heuristic were provided in [17]. Focusing on the special case where one seeks the lowest rank matrix in an affine subspace, the authors provide a "restricted isometery" condition on the linear map defining the affine subspace which guarantees the minimum nuclear norm solution is the minimum rank solution. Moreover, they provide several ensembles of affine constraints where this sufficient condition holds with overwhelming probability. This work builds on seminal developments in "compressed sensing" that develop conditions for when minimizing the ℓ_1 norm of a vector over an affine space returns the sparsest vector in that space (see, e.g., [5], [4], [2]). There is a strong parallelism between the sparse approximation and rank minimization settings. The rank of a diagonal matrix is equal to the number of non-zeros on the diagonal. Similarly, the sum of the singular values of a diagonal matrix is equal to the ℓ_1 norm of the diagonal. Exploiting the parallels, the authors in [17] were able to extend much of the analysis developed for the ℓ_1 heuristic to provide guarantees for the nuclear norm heuristic.

Building on a different collection of developments in compressed sensing [6], [7], [19], in this paper we present a *necessary* and sufficient condition for the solution of the nuclear norm heuristic to coincide with the minimum rank solution in an affine space. The condition characterizes a particular property of the null-space of the linear map which defines the affine space. To demonstrate why this result is of practical use to the controls community, we also present a reduction of the standard Linear Matrix Inequality (LMI) constrained rank minimization problem to a rank minimization problem with only equality constraints. Moreover, we show that when the linear map defining the constraint set is generated by sampling its entries independently from a Gaussian distribution, the null-space characterization holds with overwhelming probability provided the dimensions of the equality constraints are of appropriate size. We provide numerical experiments that demonstrate that even when

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matrix dimensions are small, the nuclear norm heuristic does indeed always recover the minimum rank solution when the number of constraints is sufficiently large.

II. NOTATION AND PRELIMINARIES

For a rectangular matrix $X \in \mathbb{R}^{n_1 \times n_2}$, X^* denotes the transpose of X. $\mathrm{vec}(X)$ denotes the vector in $\mathbb{R}^{n_1 n_2}$ with the columns of X stacked on top of one and other, and $\mathrm{mat}(v)$ for $v \in \mathbb{R}^{n_1 n_2}$ denotes the inverse of vec.

$\sigma_i(X)$ denotes the i-th largest singular value of X and is equal to the square-root of the i-th largest eigenvalue of XX^*. The rank of X will usually be denoted by r, and is equal to the number of nonzero singular values. For matrices X and Y of the same dimensions, we define the inner product in $\mathbb{R}^{n_1 \times n_2}$ as $\langle X, Y \rangle := \mathrm{trace}(X^* Y) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} X_{ij} Y_{ij}$. The norm associated with this inner product is called the Frobenius (or Hilbert-Schmidt) norm $||\cdot||_F$. The Frobenius norm is also equal to the Euclidean, or ℓ_2, norm of the vector of singular values, i.e.,

$$\|X\|_F := \left(\sum_{i=1}^{r} \sigma_i^2 \right)^{\frac{1}{2}} = \sqrt{\langle X, X \rangle} = \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} X_{ij}^2 \right)^{\frac{1}{2}}$$

The operator norm (or induced 2-norm) of a matrix is equal to its largest singular value (i.e., the ℓ_∞ norm of the singular values):

$$\|X\| := \sigma_1(X).$$

The nuclear norm of a matrix is equal to the sum of its singular values, i.e.,

$$\|X\|_* := \sum_{i=1}^{r} \sigma_i(X)\,,$$

and is alternatively known by several other names including the Schatten 1-norm, the Ky Fan r-norm, and the trace class norm. These three norms are related by the following inequalities which hold for any matrix X of rank at most r:

$$||X|| \leq ||X||_F \leq ||X||_* \leq \sqrt{r}||X||_F \leq r||X||. \qquad (1)$$

We also state the following easily verified fact that will be used extensively throughout.

Lemma 2.1: Suppose X and Y are $n_1 \times n_2$ matrices such that $X^*Y = 0$ and $XY^* = 0$. Then $\|X + Y\|_* = \|X\|_* + \|Y\|_*$.

Indeed, if $X^*Y = 0$ and $XY^* = 0$, we can find a coordinate system in which

$$X = \left\| \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right\|_* \quad \text{and} \quad Y = \left\| \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} \right\|_*$$

from which the lemma trivially follows.

III. MAIN RESULTS

Let X be an $n_1 \times n_2$ matrix decision variable. Without loss of generality, we will assume throughout that $n_1 \leq n_2$. Let $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m$ be a linear map, and let $b \in \mathbb{R}^m$. The main optimization problem under study is

$$\begin{array}{ll} \text{minimize} & \mathrm{rank}(X) \\ \text{subject to} & \mathcal{A}(X) = b, \end{array} \qquad (2)$$

As described in the introduction, our main concern is when the optimal solution of (2) coincides with the optimal solution of

$$\begin{array}{ll} \text{minimize} & \|X\|_* \\ \text{subject to} & \mathcal{A}(X) = b, \end{array} \qquad (3)$$

Whenever $m < n_1 n_2$, the null space of $\mathcal{A}$, that is the set of Y such that $\mathcal{A}(Y) = 0$, is not empty. The following theorem finds a critical property of this null space that guarantees when the nuclear norm heuristic succeeds. Our main result is the following

Theorem 3.1: Let X_0 be the optimal solution of (2) and X_* as the optimal solution of (3). Assume that X_0 has rank $r < n_1/2$. Then

1) If for every Y in the null space of $\mathcal{A}$ and for every decomposition
$$Y = Y_1 + Y_2,$$
where Y_1 has rank r and Y_2 has rank greater than r, it holds that
$$\|Y_1\|_* < \|Y_2\|_*,$$
then $X_* = X_0$.
2) Conversely, if the condition of part 1 is not true, then there exists a vector $b \in \mathbb{R}^m$ and a rank r X_0 such that $X_* \neq X_0$.

This result is of interest for multiple reasons. First, as shown in Section V, many of the rank minimization problems of interest to the controls community can be written in the form of (2). To be precise, we have the following

Theorem 3.2: Let $\mathcal{M}$ be a linear map of $a \times b$ matrices into $\mathbb{R}^c$ and $\mathcal{C}$ maps $a \times b$ matrices into symmetric $d \times d$ matrices. Then the LMI constrained rank minimization problem

$$\begin{array}{ll} \text{minimize} & \mathrm{rank}(X) \\ \text{subject to} & \mathcal{M}(X) = b \\ & \mathcal{C}(X) \succeq 0 \end{array}$$

can be equivalently formulated as

$$\begin{array}{ll} \text{minimize} & \mathrm{rank}(X) + \lambda\, \mathrm{rank}(Z) \\ \text{subject to} & \mathcal{A}(X) = b \\ & Z = \begin{bmatrix} I_d & B \\ B^* & \mathcal{C}(X) \end{bmatrix} \end{array}$$

for any $\lambda > a$. Note that in this is a formulation with a $(a + 2d) \times (b + 2d)$ dimensional decision variable and a linear map into $c + 2d(a + b) + d^2 - \frac{d}{2}$ dimensions.

Secondly, in Section VI we present a distribution over instances of (2) where the conditions of Theorem 3.1 hold with overwhelming probability. Note that for a linear map $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m$, we can always find an $m \times n_1 n_2$ matrix $\mathbf{A}$ such that

$$\mathcal{A}(X) = \mathbf{A}\, \mathrm{vec}\, X\,. \qquad (4)$$

In the case where $\mathbf{A}$ has entries sampled independently from a zero-mean, unit variance Gaussian distribution, then the null space characterization of theorem 3.1 holds with overwhelming probability provided m is large enough. The particular details describing the relationship between the dimensions of the decision variable, the rank of the optimal

solution, and the number of equations are described in detail in Section VI.

IV. NECESSARY AND SUFFICIENT CONDITIONS

We first prove our necessary and sufficient condition for success of the nuclear norm heuristic. We will need the following technical lemma which allows us to exploit Lemma 2.1 in our proof.

Lemma 4.1: Let X be an $n_1 \times n_2$ with rank rank $r < \frac{n_1}{2}$ and Y be an arbitrary $n_1 \times n_2$ matrix. Let P_X^c and P_X^r be the matrices that project onto the column and row spaces of X respectively. Then if $P_X^r Y P_X^c$ has full rank, Y can be decomposed as

$$Y = Y_1 + Y_2, \tag{5}$$

where Y_1 has rank r, and

$$\|X + Y_2\|_* = \|X\|_* + \|Y_2\|_*. \tag{6}$$

Proof: Without loss of generality, we can write X as

$$X = \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix},$$

where X_{11} is $r \times r$ and full rank. Accordingly, Y becomes

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix},$$

where Y_{11} is full rank since $P_X^r Y P_X^c$ is. The decomposition is now clearly

$$Y = \underbrace{\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{21}Y_{11}^{-1}Y_{12} \end{bmatrix}}_{Y_1} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & Y_{22} - Y_{21}Y_{11}^{-1}Y_{12} \end{bmatrix}}_{Y_2}.$$

That Y_1 has rank r follows from the fact that the rank of a block matrix is equal to the rank of a diagonal block plus the rank of its Schur complement (see, e.g., [11, §2.2]). That $\|X_1 + Y_2\|_* = \|X_1\|_* + \|Y_2\|*$ follows from Lemma 2.1. ■

We can now provide a proof of Theorem 3.1.

Proof: We begin by proving the converse. Assume the condition of part 1 is violated, i.e., there exists some Y, such that $\mathcal{A}(Y) = 0$, $Y = Y_1 + Y_2$, $\mathrm{rank}(Y_2) > \mathrm{rank}(Y_1) = r$, yet $\|Y_1\|_* > \|Y_2\|_*$. Now take $X_0 = Y_1$ and $b = \mathcal{A}(X_0)$. Clearly, $\mathcal{A}(-Y_2) = b$ (since Y is in the null space) and so we have found a matrix of higher rank, but lower nuclear norm.

For the other direction, assume the condition of part 1 holds. Now use Lemma 4.1 with $X = X_0$ and $Y = X_* - X_0$. That is, let P_X^c and P_X^r be the matrices that project onto the column and row spaces of X_0 respectively and assume that $P_{X_0}^r(X_* - X_0)P_{X_0}^c$ has full rank. Write $X_* - X_0 = Y_1 + Y_2$ where Y_1 has rank r and $\|X_0 + Y_2\|_* = \|X_0\|_* + \|Y_2\|_*$. Assume further that Y_2 has rank larger than r (recall $r < n/2$). We will consider the case where $P_{X_0}^r(X_* - X_0)P_{X_0}^c$ does not have full rank and/or Y_2 has rank less than or equal to r in the appendix. We now have:

$$\begin{aligned} \|X_*\|_* &= \|X_0 + X_* - X_0\|_* \\ &= \|X_0 + Y_1 + Y_2\|_* \\ &\geq \|X_0 + Y_2\|_* - \|Y_1\|_* \\ &= \|X_0\|_* + \|Y_2\|_* - \|Y_1\|_* \quad \text{by Lemma 4.1} \end{aligned}$$

But $\mathcal{A}(Y_1 + Y_2) = 0$, so $\|Y_2\|_* - \|Y_1\|_*$ non-negative and therefore $\|X_*\|_* \geq \|X_0\|_*$. Since X_* is the minimum nuclear norm solution, implies that $X_0 = X_*$. ■

For the interested reader, the argument for the case where $P_{X_0}^r(X_* - X_0)P_{X_0}^c$ does not have full rank or Y_2 has rank less than or equal to r can be found in the appendix.

V. REDUCTION TO THE AFFINE CASE

The preceding result only analyzes the affine rank minimization problem and do not extend to the case of arbitrary convex constraints. However, the affine case is far more general than it appears at first glance. For example, we can again use the fact that the rank of a block symmetric matrix is equal to the rank of a diagonal block plus the rank of its Schur complement to cast any LMI in X as a rank constraint. Indeed, given $\mathcal{C}(X) \in \mathcal{S}^{d\times d}$, its positive semidefiniteness can be equivalently expressed through a rank constraint, since $\mathcal{C}(X) \succeq 0$ if and only if

$$\mathrm{rank}\left(\begin{bmatrix} I_d & B \\ B^* & \mathcal{C}(X) \end{bmatrix}\right) = d$$

for some $B \in \mathbb{R}^{d\times d}$. That is, if there exist matrices X and B satisfying the equality above, then $f(X) = B^*B \succeq 0$. We can also impose the rank constraint $\mathrm{rank}(\mathcal{C}(X)) \leq r$ by choosing B to be of size $r \times d$ and having I_r in the $(1,1)$ block. Certainly, this is not an efficient way to solve standard LMIs for which polynomial time algorithms already exist, but this example allows us to reformulate rank constrained LMIs as linearly constrained LMIs and may allow us to characterize for which LMIs the nuclear norm heuristic succeeds.

Consider the LMI constrained rank minimization problem

$$\begin{array}{ll} \text{minimize} & \mathrm{rank}(X) \\ \text{subject to} & \mathcal{A}(X) = b \\ & C(X) \succeq 0 \end{array} \tag{7}$$

where X, the decision variable is an $a \times b$ matrix (without loss of generality, $a \leq b$), $\mathcal{A}$ is some linear map of $a \times b$ matrices into $\mathbb{R}^c$ and $\mathcal{C}$ maps $a \times b$ matrices into symmetric $d \times d$ matrices. We can reformulate this problem into affine form by noting that is equivalent to

$$\begin{array}{ll} \text{minimize} & \mathrm{rank}(X) + \lambda\,\mathrm{rank}(Z) \\ \text{subject to} & \mathcal{A}(X) = b \\ & Z = \begin{bmatrix} I_d & B \\ B^* & \mathcal{C}(X) \end{bmatrix} \end{array} \tag{8}$$

for any $\lambda > a$. Note that by dimension counting, the block diagonal decision variable in (8) is $(a + 2d) \times (b + 2d)$. Also, by constraint counting, we see that there are a total of $c + 2d(a + b) + d^2 - \frac{d}{2}$ equations needed to both specify $\mathcal{A}(X) = b$ and to define Z.

The equivalence can be seen as follows. Let p_1^* denote the optimum of (7) and p_2^* denote the optimum of (8). Certainly, for any optimal solution X of (7), we can factor $\mathcal{C}(X) = B^*B$ to construct a Z with rank d, implying $p_1^* + \lambda d \geq p_2^*$.

Conversely, let X_0 be an optimal solution for (7). Then there exists a $B \in \mathbb{R}^{d\times d}$ such that

$$Z_0 := \begin{bmatrix} I_d & B_0 \\ B_0^* & \mathcal{C}(X_0) \end{bmatrix} \succeq 0\,.$$

That is, the pair (X_0, Z_0) is feasible for (8). Let X_1 and Z_1 be feasible for (8) and suppose $\operatorname{rank}(Z_1) > d$. Then

$$\begin{aligned} \operatorname{rank}(X_1) + \lambda \operatorname{rank}(Z_1) &\geq (\operatorname{rank}(X_1) + \lambda) + \lambda \operatorname{rank}(Z_0) \\ &\geq \operatorname{rank}(X_0) + \lambda \operatorname{rank}(Z_0) \\ &= p_1^* + \lambda d\,. \end{aligned} \tag{9}$$

Now, if (X_2, Z_2) were feasible for (8) and $\operatorname{rank}(Z_2) = d$, then X_2 would be feasible for (7) and would have rank at most p_1^*. Therefore (X_0, Z_0) is an optimal solution for (8). Note that if we have an upper bound on the rank of the optimal X for (7), then the same argument reveals that any λ greater than that *a priori* rank bound will also suffice to guarantee that (7) and (8) have the same optimal solutions.

Using this equivalence, we may apply the analysis tools developed here to determine if the minimum rank solution is found when using nuclear norm heuristic.

VI. PROBABILISTIC GENERATION OF CONSTRAINTS SATISFYING NULL-SPACE CHARACTERIZATION

We now present a family of random equality constraints under which the nuclear norm heuristic succeeds with overwhelming probability. For simplicity of notation in the proofs, we consider the case of square matrices. These results can be then translated into rectangular matrices by padding with rows/columns of zeros to make the matrix square.

Theorem 6.1: Let X be an arbitrary real $n\times n$ matrix with rank $r < n/2$ and suppose $\mathcal{A}$ is defined as

$$\mathcal{A}(X) = \mathbf{A}\operatorname{vec} X\,, \tag{10}$$

where $\mathbf{A}$ is a $m \times n^2$ matrix with entries independently chosen from a Gaussian with zero-mean and unit variance. Let $b = \mathbf{A}\operatorname{vec} X$ and let X_* denote the solution to the nuclear norm minimization problem, i.e.,

$$X_* = \arg \min_{\mathcal{A}(\hat{X})=b} \|\hat{X}\|_*.$$

Fix $\beta = \frac{r}{n} \leq \frac{1}{2}$ and $\mu = \frac{m}{n^2}$. Then, X_* coincides with X, for all X of rank r with probability exceeding

$$1 - 12n^2(1-\mu)\exp\left(-n^2\left[h(\beta,\mu) - (1-\mu)\log(n)\right]\right)$$

where

$$\begin{aligned} h(\beta,\mu) = \sup_{\epsilon_1,\epsilon_2} \Bigg\{ & \frac{(1-\beta)^2 f^2(1-\beta)}{2(1+\frac{2}{1-2\epsilon_1}\sqrt{\frac{\beta}{1-\beta}})^2} g^2(\epsilon_1,\epsilon_2,\beta) \\ & - \beta(2\beta-1)\log\left(\tfrac{c_S}{\epsilon_1}\right) \\ & -(1-\mu)\log\left(\tfrac{3\sqrt{1-\mu}}{\epsilon_2}\right)\Bigg\}\,, \end{aligned}$$

$$\begin{aligned} g(\epsilon_1,\epsilon_2,\beta) = & \frac{1-\epsilon_2}{1+\epsilon_2} - \frac{\beta}{\sqrt{1-\beta}}\cdot\frac{f(\frac{\beta}{1-\beta})}{f(1-\beta)} \\ & - \frac{1+2\epsilon_1}{1-2\epsilon_1}\cdot\frac{\beta^{3/2}f(1)}{1-\beta f(1-\beta)}\,, \end{aligned}$$

and

$$\begin{aligned} f(\gamma) &= \frac{1}{2\pi\gamma}\int_{s_1}^{s_2}\sqrt{\frac{(z-s_1)(s_2-z)}{z}}dz \\ s_1 &= (1-\sqrt{\gamma})^2 \\ s_2 &= (1+\sqrt{\gamma})^2 \end{aligned}$$

and where c_S is the covering constant of the Stiefel manifold of r dimensional projections in n dimensional space.

As one can deduce from the complicated formulae in the statement of this theorem, the proof of this result is rather technical. However, the argument is straightforward. We note that if $\mathbf{A}$ is Gaussian, then its null-space can be assumed to be Gaussian. Using this observation, we show that for a fixed element of the null-space and a fixed low-rank matrix, the probability that the null-space characterization fails to hold is exponentially small in n. We then union bound over a covering of all rank r matrices and all parameterizations of the null space to provide the desired result. A full outline is provided in the appendix, but some details had to be omitted for space constraints.

There are several consequences we would like to note about this somewhat intricate probability estimate. First, note that as n goes to infinity, as long as $h(\beta,\mu)$ is positive, then the heuristic succeeds with overwhelming probability. For large n, the conditions for success are for the most part only a function of the parameters β and μ as n only enters as logarithmic factors. As we will show in section VII, this independence of n appears to hold for very small n, and our asymptotic estimates are somewhat conservative. Also, looking at the constraint set for ϵ_1 and ϵ_2 gives us a region for β that can be recovered by nuclear norm relaxation.

Corollary 6.2: Nuclear norm relaxation can, with overwhelming probability over $\mathcal{A}(\cdot)$, recover any X of rank r, when n is large, provided

$$1 - \frac{\beta}{\sqrt{1-\beta}}\cdot\frac{f(\frac{\beta}{1-\beta})}{f(1-\beta)} - \frac{\beta^{3/2}f(1)}{1-\beta f(1-\beta)} > 0.$$

VII. NUMERICAL EXPERIMENTS

To demonstrate that the nuclear heuristic succeeds in non-asymptotic scenarios, we conducted a series of experiments for a variety of the matrix sizes n, ranks r, and numbers of measurements m. As in the previous section, we let $\beta = \frac{r}{n}$ and $\mu = \frac{m}{n^2}$. For a fixed n, we constructed random recovery scenarios for low-rank $n \times n$ matrices. For each n, we varied μ between 0 and 1 where the matrix is completely determined. For a fixed n and μ, we generated all possible ranks such that $\beta(2-\beta) \leq \mu$. This cutoff was chosen because beyond that point there would be an infinite set of matrices of rank r satisfying the m equations.

For each (n,μ,β) triple, we repeated the following procedure 10 times. A matrix of rank r was generated by choosing

two random $n \times r$ factors Y_L and Y_R with i.i.d. random entries and setting $Y_0 = Y_L Y_R^*$. A matrix $\mathbf{A}$ was sampled from the Gaussian ensemble with m rows and n^2 columns. Then the nuclear norm minimization

$$\begin{array}{ll} \text{minimize} & \|X\|_* \\ \text{subject to} & \mathbf{A}\,\mathrm{vec}\,X = \mathbf{A}\,\mathrm{vec}\,Y_0 \end{array} \tag{11}$$

was solved using the freely available software SeDuMi [20]. On a 2.0 GHz Laptop, each semidefinite program could be solved in less than two minutes for 40×40 dimensional X. We declared Y_0 to be recovered if $\|X - Y_0\|_F / \|Y_0\|_F < 10^{-3}$. Figure 1 shows the results of these experiments for $n = 30$ and 40. The color of the cell in the figures reflects the empirical recovery rate of the 10 runs (scaled between 0 and 1). White denotes perfect recovery in all experiments, and black denotes failure for all experiments. It is remarkable to note the similarity between the experiments for $n = 30$ and $n = 40$. There is a strong indication that the recovery conditions are only a function of β and μ as n grows.

VIII. CONCLUSIONS AND FUTURE WORKS

We have presented a necessary and sufficient condition for the nuclear norm heuristic (and hence also the trace heuristic) to find the lowest rank solution of an affine set, and also shown how to reformulate LMI constrained rank minimization problems in the affine form. It would be interesting to directly formulate necessary and sufficient conditions for the LMI constrained rank minimization problem (7) that do not require such an affine reformulation. Along the same lines, it would be interesting to provide random instances of LMI constrained rank minimization problems that satisfy such necessary and sufficient conditions with high probability.

Future work should also investigate if the probabilistic analysis that provides the bounds in Theorem 6.1 can be tightened at all. This could be achieved via a direct Chernoff bound rather than the union bound applied in the proof that we sketch in the appendix. In particular, in the related work in compressed sensing [19], the authors provide conditions for the success of the ℓ_1 heuristic in cardinality minimization that are independent of the dimension of the vector. Finding such dimension-free guarantees for rank minimization is an intriguing open problem.

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APPENDIX

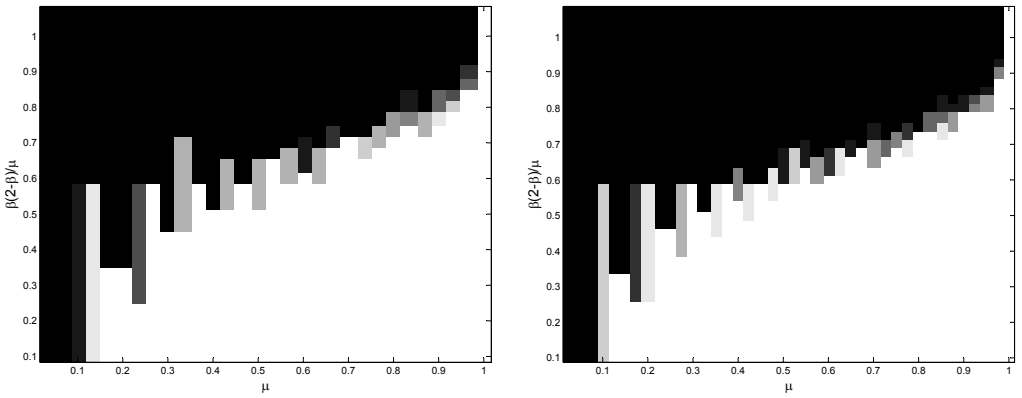
A. Rank-deficient case of Theorem 3.1

As promised above, here is the completion of the proof of Theorem 3.1 *Proof:* In an appropriate basis, we may write

$$X_0 = \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad X_* - X_0 = Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$$

If Y_{11} and $Y_{22} - Y_{21}Y_{11}^{-1}Y_{12}$ have full rank, then all our previous arguments apply. Thus, assume that at least one of them is not full rank. Nonetheless, it is always possible to find an *arbitrarily small* $\epsilon > 0$ such that

$$Y_{11} + \epsilon I \quad \text{and} \quad \begin{bmatrix} Y_{11} + \epsilon I & Y_{12} \\ Y_{21} & Y_{22} + \epsilon I \end{bmatrix}$$



(a) (b)

Fig. 1. For each (n, μ, β) triple, we repeated the following procedure ten times. A matrix of rank $r = \beta n$ was generated by choosing two random $n \times r$ factors Y_L and Y_R with i.i.d. random entries and set $Y_0 = Y_L Y_R^*$. We select a matrix $\mathbf{A}$ from the Gaussian ensemble with $m = \mu n^2$ rows and n^2 columns. Then we solve the nuclear norm minimization subject to $\mathbf{A}\,\mathrm{vec}\,X = \mathbf{A}\,\mathrm{vec}\,Y_0$ We declare Y_0 to be recovered if $\|X - Y_0\|_F / \|Y_0\|_F < 10^{-3}$. The results are shown for (a) $n = 30$ and (b) $n = 40$. The color of each cell reflects the empirical recovery rate (scaled between 0 and 1). White denotes perfect recovery in all experiments, and black denotes failure for all experiments.

are full rank. This, of course, is equivalent to having $Y_{22} + \epsilon I - Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12}$ full rank. We can write

$$\begin{aligned}
\|X_*\|_* &= \|X_0 + X_* - X_0\|_* \\
&= \left\| \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \right\|_* \\
&\geq \left\| \begin{bmatrix} X_{11} - \epsilon I & 0 \\ 0 & Y_{22} - Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12} \end{bmatrix} \right\|_* \\
&\quad - \left\| \begin{bmatrix} Y_{11} + \epsilon I & Y_{12} \\ Y_{21} & Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12} \end{bmatrix} \right\|_* \\
&= \|X_{11} - \epsilon I\|_* \\
&\quad + \left\| \begin{bmatrix} 0 & 0 \\ 0 & Y_{22} - Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12} \end{bmatrix} \right\|_* \\
&\quad - \left\| \begin{bmatrix} Y_{11} + \epsilon I & Y_{12} \\ Y_{21} & Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12} \end{bmatrix} \right\|_* \\
&\geq \|X_0\|_* - r\epsilon \\
&\quad + \left\| \begin{bmatrix} \epsilon I - \epsilon I & 0 \\ 0 & Y_{22} - Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12} \end{bmatrix} \right\|_* \\
&\quad - \left\| \begin{bmatrix} Y_{11} + \epsilon I & Y_{12} \\ Y_{21} & Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12} \end{bmatrix} \right\|_* \\
&\geq \|X_0\|_* - 2r\epsilon \\
&\quad + \left\| \begin{bmatrix} -\epsilon I & 0 \\ 0 & Y_{22} - Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12} \end{bmatrix} \right\|_* \\
&\quad - \left\| \begin{bmatrix} Y_{11} + \epsilon I & Y_{12} \\ Y_{21} & Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12} \end{bmatrix} \right\|_* \\
&\geq \|X_0\|_* - 2r\epsilon,
\end{aligned}$$

where the last inequality follows from the condition of part 1 and noting that

$$\begin{aligned}
X_0 - X_* = &\begin{bmatrix} -\epsilon I & 0 \\ 0 & Y_{22} - Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12} \end{bmatrix} \\
&+ \begin{bmatrix} Y_{11} + \epsilon I & Y_{12} \\ Y_{21} & Y_{21}(Y_{11} + \epsilon I)^{-1} Y_{12} \end{bmatrix},
\end{aligned}$$

lies in the null space of $\mathcal{A}(\cdot)$ and the first matrix above has rank more than r. But, since ϵ can be arbitrarily small, this implies that $X_0 = X_*$. ■

B. Probabilistic analysis of null-space characterization

Proof: [Outline of Proof of Theorem 6.1]

1) We first note that the distribution of the nullspace of $\mathbf{A}$, whose entries are iid, zero-mean, unit variance, Gaussian, is given by $n \times n$ matrices Ω, such that
$$\mathrm{vec}(\Omega) = Zv,$$
where Z is a $n^2 \times (n^2 - m)$ matrix with iid $\mathcal{N}(0, 1)$ entries and v is a unit (Euclidean) norm vector, $\|v\| = 1$. Let $G^{(i)}$ denote mat applied to the ith column of Z. Throughout we will let $M := n^2 - m$
2) Using our if and only if condition for the recovery of any X of rank r, the probability that a randomly chosen A does not satisfy this condition is given by
$$\begin{aligned}
P_f = \Pr\big(&\exists P_u, P_v, \|v\| = 1 \\
&\text{s.t. } \|P_u \Omega P_v\|_* > \|\Omega - P_u \Omega P_v\|_*, \\
&\mathrm{vec}(\Omega) = Zv\big)
\end{aligned}$$
where P_u and P_v are $r \times n$ and $n \times r$ projection matrices, respectively, and $v \in \mathcal{R}^M$.
3) Using the fact that
$$\begin{aligned}
\|Pu\Omega Pv\|_* \leq &\|P_{u'} \Omega P_{v'}\|_* + \|P_u - P_{u'}\| \cdot \|\Omega\|_* \\
&+ \|P_v - P_{v'}\| \cdot \|\Omega\|_*,
\end{aligned}$$
we can cover the two Stiefel mainfolds with balls of radius ϵ_1 (in the operator norm) to obtain
$$\begin{aligned}
P_f \leq N_{\epsilon_1} \Pr\big(&\exists v,\ \|v\| = 1 \\
&\text{s.t. } \tfrac{1+2\epsilon_1}{1-2\epsilon_1} \left\| \begin{pmatrix} \Omega_{11} & 0 \\ 0 & 0 \end{pmatrix} \right\|_* > \left\| \begin{pmatrix} 0 & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix} \right\|_* \\
&\mathrm{vec}(\Omega) = Zv\big)
\end{aligned}$$
where $N_{\epsilon_1} = (c_S/\epsilon_1)^{2r(2n-r-1)/2} = (c_S/\epsilon_1)^{r(2n-r-1)}$, since the Stiefel manifolds are

each of dimension $r(2n-r-1)/2$, and where c_S is the corresponding covering constant (see, e.g., [21], [22]). Note also that the partition is such that $\|\Omega_{11}\| \in \mathcal{R}^{r\times r}$, $\|\Omega_{12}\| \in \mathcal{R}^{r\times(n-r)}$, $\|\Omega_{21}\| \in \mathcal{R}^{(n-r)\times r}$ and $\|\Omega_{22}\| \in \mathcal{R}^{(n-r)\times(n-r)}$

4) Now note that

$$\begin{aligned}\|\sum_i v_i G^{(i)}\|_* &\le \|\sum_i v_i' G^{(i)}\|_* \\ &\quad + \|v-v'\|_1 \max_{1\le i\le M}\|G^{(i)}\|_* \\ &\le \|\sum_i v' G^{(i)}\|_* \\ &\quad + \sqrt{M}\|v-v'\| \max_{1\le i\le M}\|G^{(i)}\|_*,\end{aligned}$$

where $\|v\|_1$ denotes the ℓ_1 norm of v, i.e., the sum of the absolute values of the entries of v. We can cover the M-dimensional unit sphere with balls of radius ϵ_2 to obtain

$$\begin{aligned}P_f \le N_{\epsilon_1}N_{\epsilon_2}\times& \\ \Pr\Bigg(&\tfrac{1+2\epsilon_1}{1-2\epsilon_1}\|\Omega_{11}\|_* > \left\|\begin{pmatrix}0 & \Omega_{12}\\ \Omega_{21} & \Omega_{22}\end{pmatrix}\right\|_* \\ &- \epsilon_2\tfrac{1+2\epsilon_1}{1-2\epsilon_1}\max_{1\le i\le M}\left\|G_{11}^{(i)}\right\|_* \\ &-\epsilon_2\max_{1\le i\le M}\left\|\begin{pmatrix}0 & G_{12}^{(i)}\\ G_{21}^{(i)} & G_{22}^{(i)}\end{pmatrix}\right\|_*\Bigg),\end{aligned}$$

where $N_{\epsilon_2} = (\frac{3\sqrt{M}}{\epsilon_2})^M$. This latter can be further bounded as

$$\begin{aligned}P_f \le N_{\epsilon_1}N_{\epsilon_2}\Pr\Bigg(&\frac{1+2\epsilon_1}{1-2\epsilon_1}\|\Omega_{11}\|_* + \|\Omega_{21}\|_* \\ &+ \epsilon_2\frac{1+2\epsilon_1}{1-2\epsilon_1}\max_{1\le i\le M}\|G_{11}^{(i)}\|_* \\ &+ \epsilon_2\max_{1\le i\le M}\|G_{21}^{(i)}\|_* \\ &\epsilon_2\max_{1\le i\le M}\left\|\begin{pmatrix}G_{12}^{(i)}\\ G_{22}^{(i)}\end{pmatrix}\right\|_* > \left\|\begin{pmatrix}\Omega_{12}\\ \Omega_{22}\end{pmatrix}\right\|_*\Bigg).\end{aligned} \tag{12}$$

5) To bound the above probability, we use two facts.

a) For any constant thresholds η, η_1, η_2, η_3, and η_4, we have

$$\begin{aligned}&\Pr(a_1+a_2+\max_{1\le i\le M}a_3^{(i)} \\ &\qquad + \max_{1\le i\le M}a_4^{(i)} + \max_{1\le i\le M}a_5^{(i)} > b) \\ \le& M\Pr(a_5>\eta_4) + M\Pr(a_4>\eta_3-\eta_4) \\ &+ M\Pr(a_3>\eta_2-\eta_3) + \Pr(a_2>\eta_1-\eta_2) \\ &+ \Pr(a_1>\eta-\eta_1) + \Pr(b<\eta).\end{aligned} \tag{13}$$

We will apply the above formula to (12) with

$$\begin{cases} a_1 &= \frac{1+2\epsilon_1}{1-2\epsilon_1}\|\Omega_{11}\|_* \\ a_2 &= \|\Omega_{21}\|_* \\ a_3^{(i)} &= \epsilon_2\frac{1+2\epsilon_1}{1-2\epsilon_1}\|G_{11}^{(i)}\|_* \\ a_4^{(i)} &= \epsilon_2\|G_{21}^{(i)}\|_* \\ a_5^{(i)} &= \epsilon_2\left\|\begin{pmatrix}G_{12}^{(i)}\\ G_{22}^{(i)}\end{pmatrix}\right\|_* \\ b &= \left\|\begin{pmatrix}\Omega_{12}\\ \Omega_{22}\end{pmatrix}\right\|_*\end{cases}$$

b) The second fact is the concentration of measure for Gaussian random variables (see, e.g. [12]) which states that if F is a function of Gaussian variables with Lipshitz constant $c_{\rm Lip}$ then for $\delta>0$

$$\Pr(|F-\mathbb{E}[F]|>\delta) \le 2\exp\left(-\frac{\delta^2}{2c_{\rm Lip}^2}\right).$$

We apply this with f being the nuclear norm. Note that for an arbitrary $n_1\times n_2$ ($n_1\le n_2$) matrices W_1 and W_2,

$$\begin{aligned}\|W_1\|_* - \|W_2\|_* &\le \|W_1-W_2\|_* \\ &\le \sqrt{n_1}\|W_1-W_2\|_F\end{aligned}$$

and hence the Lipshitz constant of the nuclear norm is given by $\sqrt{n_1}$. Now suppose H is a $n_1\times n_2$ matrix with zero-mean iid entries with variance $\frac{1}{n_2}$. Denote the singular values of H by σ_i, $i=1,\ldots,n_1$. Then

$$\begin{aligned}&\Pr\left(|\|H\|_* - n_1\mathbb{E}[\sigma_i]| \ge (n_1+n_2)^{3/2}\delta\right) \\ \le& 2e^{-\frac{1+n_2/n_1}{2}(n_1+n_2)^2\delta^2}.\end{aligned} \tag{14}$$

6) To apply the above formula to the a_i's and b defined earlier, we need to know the asymptotic mean of the nuclear norm. Let H be as above. We will define

$$\nu_{n_1,n_2} = n_1\mathbb{E}\sigma_i = n_1\sqrt{n_2}f(\gamma) + \text{lower order terms},$$

where $\gamma = \frac{n_1}{n_2}$ and $f(\cdot)$ is found by integrating $\sqrt{\lambda}$ against the Marčenko-Pastur distribution (see, e.g., [14], [1]:

$$\begin{aligned}f(\gamma) &= \frac{1}{2\pi\gamma}\int_{s_1}^{s_2}\sqrt{\frac{(z-s_1)(s_2-z)}{z}}dz \\ s_1 &= (1-\sqrt{\gamma})^2 \\ s_2 &= (1+\sqrt{\gamma})^2.\end{aligned}$$

Note further that

$$\nu_{n_1,n_2} = n_2^{3/2}\gamma f(\gamma) + \text{lower order terms}, \tag{15}$$

which is what we shall henceforth use.

7) We can now go about bounding all the probabilities in (13). We need to take the thresholds proportion to $n^{3/2}$ and so we will define $\eta_i = n^{3/2}\delta_i$. Recalling that

$$\Pr(a_5 > \eta_4) \leq 2\exp\left(-n^2 \cdot \frac{(\delta_4 - \epsilon_2(1-\beta)f(1-\beta))^2}{2(1-\beta)\epsilon_2^2}\right) \quad (16)$$

$$\Pr(a_4 > \eta_3 - \eta_4) \leq 2\exp\left(-n^2 \cdot \frac{(\delta_3 - \delta_4 - \epsilon_2\beta\sqrt{1-\beta}f(\beta/(1-\beta)))^2}{2\beta\epsilon_2^2}\right) \quad (17)$$

$$\Pr(a_3 > \eta_2 - \eta_3) \leq 2\exp\left(-n^2 \cdot \frac{(\delta_2 - \delta_3 - \epsilon_2\frac{1+2\epsilon_1}{1-2\epsilon_1}\beta^{3/2}f(1))^2}{2\beta\epsilon_2^2\left(\frac{1+2\epsilon_1}{1-2\epsilon_1}\right)^2}\right) \quad (18)$$

$$\Pr(a_2 > \eta_1 - \eta_2) \leq 2\exp\left(-n^2 \cdot \frac{(\delta_1 - \delta_2 - \beta\sqrt{1-\beta}f(\beta/(1-\beta)))^2}{2\beta}\right) \quad (19)$$

$$\Pr(a_1 > \eta - \eta_1) \leq 2\exp\left(-n^2 \cdot \frac{(\delta - \delta_1 - \frac{1+2\epsilon_1}{1-2\epsilon_1}\beta^{3/2}f(1))^2}{2\beta\left(\frac{1+2\epsilon_1}{1-2\epsilon_1}\right)^2}\right) \quad (20)$$

$$\Pr(b < \eta) \leq 2\exp\left(-n^2 \cdot \frac{((1-\beta)f(1-\beta) - \delta)^2}{2(1-\beta)}\right) \quad (21)$$

TABLE I

PROBABILISTIC BOUNDS BASED ON CONCENTRATION OF MEASURE AND THE MARCHENKO-PASTUR DISTRIBUTION.

we have set $\beta = \frac{r}{n}$, some simple calculations give the bounds listed in Table 7.

8) To get the tightest bound possible in (13) we must equate the six exponents in Table 7. We have five degrees of freedom in the δ, δ_1, δ_2, δ_3 and δ_4, and so this can always be done by solving a system of linear equations. Note that the exponents above have been written in such a way that the arguments inside the "squares" must be positive.
Doing this, for example, yields

$$\begin{aligned}\delta = &\frac{1}{1+\frac{2}{1-2\epsilon_1}\sqrt{\frac{\beta}{1-\beta}}}\times \\ &\left[\sqrt{\frac{\beta}{1-\beta}}(1-\beta)f(1-\beta)\right. \\ &+\tfrac{2\epsilon_2}{1+\epsilon_2}(1-\beta)f(1-\beta)+\beta\sqrt{1-\beta}f(\frac{\beta}{1-\beta}) \\ &\left.+\tfrac{1+2\epsilon_1}{1-2\epsilon_1}\left(\beta^{3/2}f(1)+\sqrt{\beta(1-\beta)}f(1-\beta)\right)\right].\end{aligned}$$

Of course, this solution only makes sense if $(1-\beta)f(1-\beta) - \delta > 0$. Simplifying this yields

$$\tfrac{1-\epsilon_2}{1+\epsilon_2} - \tfrac{\beta}{\sqrt{1-\beta}} \cdot \tfrac{f(\frac{\beta}{1-\beta})}{f(1-\beta)} - \tfrac{1+2\epsilon_1}{1-2\epsilon_1} \cdot \tfrac{\beta^{3/2}f(1)}{1-\beta f(1-\beta)} > 0,$$

which is the constraint on $g(\cdot,\cdot,\cdot)$ in Theorem (6.1). Plugging the solution for the δ's shows that the optimal exponent is

$$-n^2\frac{(1-\beta)^2f^2(1-\beta)}{2(1+\frac{2}{1-2\epsilon_1}\sqrt{\frac{\beta}{1-\beta}})^2}g^2(\epsilon_1,\epsilon_2,\beta).$$

We thus, using our definition $m = \mu n^2$,have

$$\begin{aligned}P_f \leq &12n^2(1-\mu)\left(\tfrac{c_S}{\epsilon_1}\right)^{\beta(2\beta-1)n^2}\left(\tfrac{3n\sqrt{1-\mu}}{\epsilon_2}\right)^{(1-\mu)n^2} \\ &\times\exp\left(-n^2\frac{(1-\beta)^2f^2(1-\beta)}{2(1+\frac{2}{1-2\epsilon_1}\sqrt{\frac{\beta}{1-\beta}})^2}g^2(\epsilon_1,\epsilon_2,\beta)\right).\end{aligned}$$

This is essentially the statement of our thoerem. Note that $P_f \to 0$ if the coefficient of n^2 in the exponent is negative.

■