AN APPLICATION OF BRUN'S SIEVE

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1. Introduction

Viggo Brun developed a sieve method which he used to show that the inverse sum over all twin primes converges. This famous result led to the development of Sieve methods in detail, which helped prove a host of fascinating results. In this article we apply Brun's method in a straightforward way to show that the sum over the inverse of *Double primes* converges. We follow the treatment in [Odl71] almost without any change, which speaks for the power of even the simplest sieve argument.

2. Main Theorem

Let $D = \{q = 2p+1 \mid p, q \text{ both prime }\}$. Our main result would be to show that

$$\sum_{q \in D} \frac{1}{q} \text{ converges.}$$

Note that

$$\sum_{q \in D} \frac{1}{q} \leq \sum_{p:\text{ prime}} \frac{1}{2p+1}$$

But the sum on the right hand side diverges since:

$$\begin{aligned} \sum_{p:\text{ prime}} \frac{1}{2p+1} &\geq \sum_{p:\text{ prime}} \frac{1}{4p} \\ &\geq \frac{1}{4} \sum_{p:\text{ prime}} \frac{1}{p} \end{aligned}$$

and the inverse sum over all primes diverges. So the simple upper bounding technique did not work and we will need to do a more detailed analysis of the sum.
We will first prove the following result. Let

$$D(x) = \sum_{p \leq x, p \in D} 1.$$

Theorem 2.1.

$$D(x) = O\left(x\left(\frac{\lg\lg x}{\lg x}\right)^2\right).$$

Proof : Let $S_{x,z} = \{n \mid n \text{ odd }, n \leq x : n(2n+1) \text{ is not divisible by any prime } \leq z\}$ and $S(x,z) = |S_{x,z}|$. $S(x,z)$ is an upper bound on $D(x)$, the idea is to make z as large as possible helping us get a good upper bound for $D(x)$.

Let $P = \{p \leq z \mid p \text{ a prime }\}$. Let $r = \pi(z) = |P|$, and $\Pi = \prod_{p \in P} p$. Now a number n enters the above set iff $n \perp \Pi$.

The möbius function μ has the following nice property:

$$\sum_{d \backslash n} \mu(d) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$$

Let us set

$$s'(n) = \sum_{d \backslash \gcd(n(2n+1),\Pi)} \mu(d).$$

So $s'(n)$ operates as the characteristic function for the set $S_{x,z}$. Thus we can write

$$S(x,z) = \sum_{1 \le n \le x,\ n \text{ odd}} s'(n).$$

Let

$$B(x,d) = |\{n \le x \mid n \text{ odd },\, d \backslash n(2n+1)\}|.$$

Now

$$\begin{aligned} S(x,z) &= \sum_{1 \le n \le x,\ n \text{ odd}} \left(\sum_{d \backslash \gcd(n(2n+1),\Pi)} \mu(d) \right) \\ &= \sum_{d \backslash \Pi} \mu(d) B(x,d) \end{aligned}$$

Let $\nu(k)$ be the number of the distinct prime divisors of k.
Note : $\mu(d) = (-1)^{\nu(d)}$.

Define

$$s(n) = \sum_{d \backslash \gcd(n(2n+1),\Pi), \nu(d) \le m} \mu(d)$$

where m is an integer we shall pick later.
We would like to show that $s(n)$ is an upper bound on $s'(n)$ and since the above quantity is easier to analyze we can get an upper bound on $S(x,z)$.
Claim: $\forall n : s'(n) \le s(n)$.
Proof of Claim: Let $t = \gcd(n(2n+1), \Pi)$. Then we have

$$s'(n) = \sum_{d \backslash t} \mu(d) \text{ and}$$

$$s(n) = \sum_{d \backslash t, \nu(d) \le m} \mu(d).$$

If $t = 1$ then $s'(n) = 1 = s(n)$.
If $t > 1$ then $s'(n) = 0$, so it suffices to show that $s(n) \ge 0$ when $t > 1$ to prove the claim.

$[t > 1]$. Since t divides Π, all divisors of t are squarefree and so using the above observation we write:

$$s(n) = \sum_{d \backslash t,\ \nu(d) \le m,\ \nu(d) \text{ even}} 1 - \sum_{d \backslash t,\ \nu(d) \le m,\ \nu(d) \text{ odd}} 1.$$

We need to show that the first sum is greater than the second. Fix a prime p which divides t. Suppose $\delta \backslash t, \nu(d) \le m$ and suppose $\nu(\delta)$ is odd, so that it enters the second sum. Now $1 \le \nu(\delta) \le m - 1$ since m is even.
Let

$$\delta' = \begin{cases} p\delta & \text{if } p \not\backslash \delta \\ \frac{\delta}{p} & \text{if } p \backslash \delta. \end{cases}$$

Now

$$\nu(\delta') = \begin{cases} \nu(\delta) + 1 & \text{if } p \not\backslash \delta \\ \nu(\delta) - 1 & \text{if } p \backslash \delta. \end{cases}$$

So that $\nu(\delta')$ is even and $0 \le \nu(\delta') \le m$. Since the correspondence between δ and δ' is one-one, there are at least as many positive terms in the first sum as there are negative terms in the second sum. Thus $s(n) \ge 0$

and we have proved the claim. ⋆

Applying the claim we have

$$\begin{aligned} S(x,z) &\le \sum_{1\le n\le x, n \text{ odd}} s(n) \\ &= \sum_{1\le n\le x, n \text{ odd}} \Bigg(\sum_{d\backslash \gcd(n(2n+1),\Pi), \nu(d)\le m} \mu(d) \Bigg) \\ &= \sum_{d\backslash \Pi, \nu(d)\le m} \mu(d) B(x,d). \end{aligned}$$

Let $\rho^{(f)}$ denote the divisors d of Π with $\nu(d) = f$ (with $\rho^{(0)} = 1$). When we use $\sum_{\rho^{(f)}}$ we will mean a sum over all the divisors satisfying the above condition.
Using this notation we have:

$$S(x,z) \le \sum_{0\le f\le m} (-1)^f \sum_{\rho^{(f)}} B(x, \rho^{(f)}).$$

Let us find a bound for $B(x,\rho^{(f)}) = B(x, p_{i_1}\cdots p_{i_f})$.
$B(x, p_{i_1}\cdots p_{i_f}) = |\{n \text{ odd} \mid n\le x, n(2n+1)\equiv 0 \mod p_{i_1}\cdots p_{i_f}\}|$. Let P_1, P_2 be a partition of $\{p_{i_1},\cdots,p_{i_f}\}$.
Since $n \perp (2n+1)$, $n(2n+1)\equiv 0 \mod p_{i_1},\cdots,p_{i_f}$ is equivalent to

$$\begin{aligned} n &\equiv 0 \mod \prod_{p\in P_1} p \\ 2n+1 &\equiv 0 \mod \prod_{p\in P_2} p \end{aligned}$$

By the Chinese Remainder Theorem, for a fixed P_1, P_2 there is exactly one solution $\mod \rho^{(f)}$. Thus for fixed P_1 and P_2 there are

$$\left\lfloor \frac{x}{2\rho^{(f)}} \right\rfloor$$

such solutions $n\le x$ since we are interested only in the odd solutions. Since there are 2^f possible choices for the sets P_1 and P_2,

$$B(x,\rho^{(f)}) = 2^f\Big(\frac{x}{2\rho^{(f)}}\Big) + 2^f\theta, |\theta|\le 1.$$

Combining this with our expansion for $S(x,z)$ we get

$$S(x,z) \le \frac{x}{2}\Bigg(\sum_{0\le f\le m} (-1)^f \sum_{\rho^{(f)}} \frac{2^f}{\rho^{(f)}}\Bigg) + \sum_{0\le f\le m} \binom{r}{f} 2^f.$$

Now

$$\sum_{0\le f\le m} \binom{r}{f} 2^f \le \sum_{0\le f\le m} 2^f r^f < (2r)^{m+1}.$$

Let $s_k(a_1,\cdots,a_t)$ be k-th elementary function of the variables $a_1,\cdots,a_t$ i.e.,

$$s_k(a_1,\cdots,a_t) = \sum_{\{i_1,\cdots,i_k\}\subseteq\{1,2,\cdots,k\}} a_{i_1}\cdots a_{i_k}.$$

Now the first term in the expression for $S(x,z)$ is

$$\begin{aligned} \sum_{0\le f\le m}\Bigg(\sum_{0\le f\le m} (-1)^f \sum_{\rho^{(f)}} \frac{2^f}{\rho^{(f)}}\Bigg) &= \sum_{0\le f\le r} (-1)^f \sum_{\rho^{(f)}} \frac{2^f}{\rho^{(f)}} + \sum_{m+1\le f\le r} (-1)^{f-1} \sum_{\rho^{(f)}} \frac{2^f}{\rho^{(f)}} \\ &= \prod_{p\le z}\Big(1-\frac{2}{p}\Big) + \sum_{m+1\le f\le r} (-1)^{f-1} s_f. \end{aligned}$$

Where $s_f = s_f\left(\frac{2}{3}, \cdots, \frac{2}{p_r}\right)$, and we have used the fact that

$$\prod_{1\le i\le n}(1 + x_i z) = \sum_{1\le i\le n} s_i z^i.$$

Thus

$$S(x,z) \le (2r)^{m+1} + \frac{x}{2}\prod_{p\le z}\left(1 - \frac{2}{p}\right) + \frac{x}{2}\sum_{m+1\le f\le r}(-1)^{f-1}s_f.$$

Since we can write a product of $(f+1)$ terms in $f+1$ ways as a product of f terms and a single factor, we have $s_1 s_f \ge (f+1)s_{f+1}$ for $f \ge 1$. This also gives us $s_2 \le \frac{s_1^2}{2}, s_3 \le \frac{s_1^3}{6}$ and in general $s_f \le \frac{s_1^f}{f!}$. If $s_1 \le (f+1)$ then $s_f \ge s_{f+1}$ by this observation.
Let us now select m (in addition to it being even) such that:

$$m + 1 \ge s_1 = \sum_{p\le z}\frac{2}{p}.$$

This forces $s_f \ge f_{f+1}$ in our analysis so that $\sum_{m+1\le f\le r}(-1)^{f-1}s_f$ becomes an alternating sum with terms of decreasing absolute magnitude.
So

$$\sum_{m+1\le f\le r}(-1)^{f-1}s_f \le s_{m+1} \le \frac{s_1^{m+1}}{(m+1)!} \le \left(\frac{es_1}{m+1}\right)^{m+1}$$

since $n! > \left(\frac{n}{e}\right)^n$.
Merten's theorem gives us:

$$s_1 = 2\sum_{p\le z}\frac{1}{p} = 2\ln\ln z + O(1).$$

Select m such that $e^2 s_1 < m + 1 < 9s_1$ for sufficiently large z. This also gives us $m < r$. Now

$$\left(\frac{es_1}{m+1}\right)^{m+1} \le \left(\frac{1}{e}\right)^{m+1} \le e^{-e^2 s_1} \le e^{-s_1}.$$

Also $1 - y \le e^{-y}$ for all real y and so

$$\prod_{p\le z}\left(1 - \frac{2}{p}\right) \le e^{-s_1}.$$

Since $r = \pi(z)$ we have $2r \le z$. Thus

$$S(x,z) \le z^{9s_1} + xe^{-s_1}.$$

By our expression for s_1 there is a constant C such that

$$2\ln\ln z - C < s_1 < 2\ln\ln z + C$$

and so for sufficiently large x and z

$$S(x,z) \le z^{18\ln\ln z + 9C} + \frac{xe^C}{(\ln z)^2}.$$

We take $z = x^{\frac{1}{20\ln\ln x}}$. Then for sufficiently large x,

$$\ln\ln z \le \ln\ln x \text{ as } \ln z = \frac{\ln x}{20\ln\ln x}.$$

This finally yields

$$S(x, x^{\frac{1}{20\ln\ln x}}) \le x^{\frac{9}{10}+o(1)} + 400e^C x\left(\frac{\ln\ln x}{\ln x}\right)^2 = O\left(x\left(\frac{\ln\ln x}{\ln x}\right)^2\right).$$

□

Theorem 2.2. $\sum_{p\in D}\frac{1}{p}$ *converges.*

Proof :

$$\sum_{p\in D}\frac{1}{p}=\sum_{2\le n}\frac{D(n)-D(n-1)}{n}$$
$$=\sum_{2\le n}D(n)\left(\frac{1}{n}-\frac{1}{n+1}\right)$$
$$=\sum_{2\le n}\frac{D(n)}{n(n+1)}.$$

By our previous theorem we have:

$$\frac{D(n)}{n(n+1)}=O\left(\frac{1}{n+1}\left(\frac{\ln\ln x}{\ln x}\right)^2\right)$$
$$=O\left(\frac{1}{n(\ln n)^{3/2}}\right)$$

and since

$$\sum_{n}\frac{1}{n(\ln n)^{3/2}}\le\int_{n}\frac{1}{n(\ln n)^{3/2}}dn$$

$$\int_2^c\frac{1}{n(\ln n)^{3/2}}dn=\frac{2}{\sqrt{\ln 2}}-\frac{2}{\sqrt{\ln c}}=O(1).$$

So the sum converges. □

References

[HR74] Halberstam H., Richert H. -E., *Sieve Methods*, Academic Press, 1974.
[Odl71] Andrew M. Odlyzko, *Sieve Methods*, California Institute of Technology (1971).