VANISHING OF COHOMOLOGY GROUPS OF AFFINE SCHEMES - (II)

DENIS XAVIER CHARLES

1. Interlude on Koszul Complexes

Assume that all rings are Noetherian.

Definition 1.1. Let R be a ring and let M be an R-module. A sequence of elements $x_1, \cdots, x_n \in R$ is called a *regular sequence* on M (or M-sequence) if

(1) $(x_1, \cdots, x_n)M \neq M$, and
(2) for $1 \leq i \leq n$, x_i is a non-zerodivisor on $M/(x_i, \cdots, x_{i-1})M$.

1.1. **Koszul complexes of length 1 and 2.** We can decide whether an element $x \in R$ is a non-zerodivisor from the homology of the complex:

$$\mathbf{K}(x) : 0 \to R \xrightarrow{x} R$$

then $H_0(\mathbf{K}(x)) = \{y \in R \ : \ yx = 0\}$, so it is zero iff x is a non-zerodivisor. This trivial remark is the basis for the homological study of regular sequences.

Let $y \in R$ be another element. Multiplication by y defines a map of complexes $\mathbf{K}(x) \to \mathbf{K}(x)$

$$\begin{tikzcd}
\mathbf{K}(x): & 0 \arrow[r] & R \arrow[r, "x"] \arrow[d, "y"'] & R . \arrow[d, "y"'] \\
\mathbf{K}(x): & 0 \arrow[r] & R \arrow[r, "x"] & R
\end{tikzcd}$$

This diagram can be used to build a larger complex as follows:

$$\begin{tikzcd}
\mathbf{K}(x,y): & 0 \arrow[r] & R \arrow[r, "x"] \arrow[dr, "y"'] & R \arrow[r] \arrow[dr, "y"] \arrow[d, phantom, "\bigoplus"] & 0 & \\
& & 0 \arrow[r] & R \arrow[r, "-x"'] & R \arrow[r] & 0
\end{tikzcd}$$

more precisely, the complex is given by

$$0 \longrightarrow R \xrightarrow[\psi = \binom{y}{x}]{} R \oplus R \xrightarrow[\varphi = (-x, y)]{} R.$$

For $r \in R$ we have $r \mapsto (yr, xr) \mapsto -xyr + yxr = 0$. We saw earlier that $H_0(\mathbf{K}(x)) = \operatorname{Ann}(x)$, now $H_0(\mathbf{K}(x,y)) = \{r \in R \ : \ rx = 0, ry = 0\}$. In particular, if x is a non-zerodivisor then $H_0(\mathbf{K}(x,y)) = 0$. Now let us study $H_1(\mathbf{K}(x,y))$. Now $\ker \varphi = \{(a,b) \in R \oplus R \ : \ -xa + yb = 0\}$, so if $(a,b) \in \ker \varphi$ then $b(y) \subseteq (x)$, in otherwords $b \in (x : y)$. Conversely, if $b \in (x : y)$ then by definition there is an $a \in R$ such that $(a,b) \in \ker \varphi$. If we assume that x is a non-zerodivisor then a is uniquely determined by b and the association $b \mapsto a$ is a module homomorphism, so the kernel is isomorphic to $(x : y)$. Now an element in

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$\mathrm{Im}\psi$ is of the form (cy, cx) so the elements of $(x : y)$ that correspond to elements of the image are the elements of (x). Thus if x is a non-zerodivisor then $H_1(\mathbf{K}(x,y)) = (x : y)/(x)$. Now $H_1(\mathbf{K}(x,y)) = 0$ iff the sequence (x, y) is a sequence satisfying condition (2) in the definition of a regular sequence.

We actually get a short exact sequence of complexes:

$$0 \to \mathbf{K}(x) \to \mathbf{K}(x,y) \to \mathbf{K}(x) \to 0.$$

This yields a long exact sequence of homology and we derive an interesting consequence by looking at a part of it:

$$\cdots \longrightarrow H_0(\mathbf{K}(x)) \xrightarrow{\delta} H_0(\mathbf{K}(x)) \longrightarrow H_1(\mathbf{K}(x,y)) \longrightarrow H_1(\mathbf{K}(x)) \longrightarrow \cdots$$

We find that the connection homomorphism δ is multiplication by y. Thus if we only assume that $H_1(\mathbf{K}(x,y)) = 0$, then $H_0(\mathbf{K}(x))/yH_0(\mathbf{K}(x)) = 0$. If we assume that R is a Noetherian local ring and that y is in the maximal ideal (i.e., not a unit) then Nakayama's lemma tells us that $H_0(\mathbf{K}(x)) = 0$! Consequently, x is a non-zerodivisor and the sequence (x, y) is regular. Thus we have proved the following theorem:

Theorem 1.2. *If R is a Noetherian local ring and x, y are in the maximal ideal, then x, y is a regular sequence iff $H_1(\mathbf{K}(x,y)) = 0$.*

Since $\mathbf{K}(x,y)$ and $\mathbf{K}(y,x)$ are isomorphic, the above theorem shows that (x, y) is a regular sequence iff (y, x) is one. By using the fact that every permutation is a product of transpositions of neighboring elements we can show the following corollary.

Corollary 1.3. *If R is a Noetherian local ring and $x_1, \cdots, x_r$ is a regular sequence of elements in the maximal ideal of R, then so is any permutation of the sequence.*

The above statement is false for non-local rings.

1.2. **General Koszul complexes.** If N is any A-module, then the exterior algebra $\wedge N$ may be defined as the free algebra $A \oplus N \oplus (N \otimes_A N) \oplus \cdots$ modulo the relations $x \otimes x = 0$ for all $x \in N$. Since $0 = (x+y) \otimes (x+y) = x \otimes x + x \otimes y + y \otimes x + y \otimes y = x \otimes y + y \otimes x$ we get that $x \otimes y = -y \otimes x$. $\wedge N$ is naturally a graded algebra and we denote the m-th homogeneous piece by $\wedge^m N$.

Now given a module N and an element $x \in N$, we define the Koszul complex to be the complex

$$\mathbf{K}(x): \qquad 0 \longrightarrow A \xrightarrow{d_x} N \xrightarrow{d_x} \wedge^2 N \xrightarrow{d_x} \cdots \xrightarrow{d_x} \wedge^i N \xrightarrow{d_x} \wedge^{i+1} N \xrightarrow{d_x} \cdots$$

where d_x sends an element a to $x \wedge a$. If N is free of rank n and $x = (x_1, \cdots, x_n) \in R^n \cong N$, then we shall write $\mathbf{K}(x_1, \cdots, x_n)$ for $\mathbf{K}(x)$.

We note the following fact (1.1.3.4 in EGA III-1).

Theorem 1.4. $H_n(\mathbf{K}(x_1, \cdots, x_n)) = A/(x_1, \cdots, x_n)$.

Proof : The right-hand end of the Koszul complex is

$$\cdots \to \wedge^{n-1} N \to \wedge^n N \to \wedge^{n+1} N = 0.$$

Let $\mathbf{e}_1, \cdots, \mathbf{e}_n$ be a basis for $N = R^n$. We have $\wedge^n N \cong R$ by sending $\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n$ to 1. Similarly $\wedge^{n-1} N \cong R^n$, with basis $\mathbf{e}_1 \wedge \cdots \wedge \hat{\mathbf{e}}_i \wedge \cdots \mathbf{e}_n$. Now the image of $\mathbf{e}_1 \wedge \cdots \wedge \hat{\mathbf{e}}_i \wedge \cdots \mathbf{e}_n$ under the differential of the Koszul complex is

$$\left(x_i \mathbf{e}_i \right) \wedge \mathbf{e}_1 \wedge \cdots \wedge \hat{\mathbf{e}}_i \wedge \cdots \mathbf{e}_n = \pm x_i \mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_n.$$

So the cokernel of the map $\wedge^{n-1}N \to \wedge^n N$ which is $H_n(\mathbf{K}(x_1, \cdots, x_n))$ is $R/(x_1, \cdots, x_n)$. $\square$

We state without proof a general statement indicating that homology of Koszul complexes is connected with lengths of maximal regular sequences.

Theorem 1.5. *Let M be a finitely generated A-module. If $H_j(M \otimes \mathbf{K}(x_1, \cdots, x_n)) = 0$ for $j < r$ while $H^r(M \otimes \mathbf{K}(x_1, \cdots, x_n)) \neq 0$, then every maximal regular sequence in $I = (x_1, \cdots, x_n) \subseteq R$ for M has length r.*

Corollary 1.6 (Proposition 1.1.4 EGA III-1)**.** *If $x_1, \cdots, x_n$ is a regular sequence for M, then $M \otimes \mathbf{K}(x_1, \cdots, x_n)$ is exact except at the right end i.e., $H_j(M \otimes \mathbf{K}(x_1, \cdots, x_n)) = 0$ for $j < n$. Furthermore, $H_n(M \otimes \mathbf{K}(x_1, \cdots, x_n)) = M/(x_1, \cdots, x_n)M$.*

Proof : The first claim follows directly from Theorem 1.5. Now writing N for A^n we observe that $H_n(M \otimes \mathbf{K}(x_1, \cdots, x_n))$ is the homology of Koszul complex at $M \otimes \wedge^{n-1}N \to M \otimes \wedge^n N \to 0$. By the right-exactness of the tensor product we have that

$$\begin{aligned}
\operatorname{coker}(M \otimes \wedge^{n-1}N \to M \otimes \wedge^n N) &= M \otimes \operatorname{coker}(\wedge^{n-1}N \to \wedge^n N) \\
&= M \otimes H_n(\mathbf{K}(x_1, \cdots, x_n)) \\
&= M \otimes R/(x_1, \cdots, x_n) \\
&= M/(x_1, \cdots, x_n)M.
\end{aligned}$$

$\square$

2. A Vanishing criterion for the Koszul Complex

Let $\mathbf{f} = (f_i) \in A^r$ and $\mathbf{g} = (g_i) \in A^r$; we use $e_\mathbf{g}$ to denote the right multiplication by the vector $\mathbf{g}$ in the exterior algebra $\wedge A^r$. The following is called the *homotopy formula*

$$i_\mathbf{f} e_\mathbf{g} + e_\mathbf{g} i_\mathbf{f} = \langle \mathbf{g}, \mathbf{f} \rangle \mathbf{1}$$

where $\mathbf{1}$ is the identity automorphism of the A-module A^r. Thus in the complex we see that this relation shows that $de_\mathbf{g} + e_\mathbf{g} d = \langle \mathbf{g}, \mathbf{f} \rangle \mathbf{1}$. Now if the ideal $(\mathbf{f}) = A$ then there is a $\mathbf{g} \in A^r$ such that $\langle \mathbf{g}, \mathbf{f} \rangle = \sum_i g_i f_i = 1$. Thus the homotopy relation shows that the identity map and the 0 map are homotopic or in other words $\mathbf{K}.(\mathbf{f})$ is homotopically trival.

Proposition 2.1. *Suppose that the ideal $(\mathbf{f})$ is equal to A. Then the complex $\mathbf{K}.(\mathbf{f})$ is homotopically trival, and thus so are the complexes $\mathbf{K}.(\mathbf{f}, M)$ and $\mathbf{K}^\cdot(\mathbf{f}, M)$.*

Corollary 2.2. *If $(\mathbf{f}) = A$ then we have that $H^\cdot(\mathbf{f}, M) = 0$ for all A-modules M.*