VANISHING OF COHOMOLOGY GROUPS OF AFFINE SCHEMES - (III)

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We preserve the notation of the previous articles.

1. The exterior algebra complex (continued)

Let $\mathbf{f} = (f_i)_{1\leq i\leq r}$ and $\mathbf{g} = (g_i)_{1\leq i\leq r}$ be two sequences of r elements from A and set $\mathbf{fg} = (f_i g_i)_{1\leq i\leq r}$. We define a map

$$\varphi_{\mathbf{g}} : \mathbf{K}.(\mathbf{fg}) \to \mathbf{K}.(\mathbf{f})$$

by extending canonically to $\wedge A^r$ the A-linear map $(x_1, \cdots, x_r) \mapsto (g_1 x_1, \cdots, g_r x_r)$ from A^r to itself.
In particular, for a pair of integers m, n with $0 \leq n \leq m$, we get maps

$$\varphi_{\mathbf{f}^{m-n}} : \mathbf{K}.(\mathbf{f}^m) \to \mathbf{K}.(\mathbf{f}^n)$$

which induce maps

$$\varphi_{\mathbf{f}^{m-n}} : \mathbf{K}^{\cdot}(\mathbf{f}^n, M) \to \mathbf{K}^{\cdot}(\mathbf{f}^m, M)$$
$$\varphi_{\mathbf{f}^{m-n}} : H^{\cdot}(\mathbf{f}^n, M) \to H^{\cdot}(\mathbf{f}^m, M)$$

(note the direction of the maps). The maps satisfy the transitive property namely, $\varphi_{\mathbf{f}^{m-p}} = \varphi_{\mathbf{f}^{m-n}} \circ \varphi_{\mathbf{f}^{n-p}}$ for $p \leq n \leq m$. This allows us to take direct limits to get

$$C^{\cdot}((\mathbf{f}), M) = \varinjlim_{n} \mathbf{K}^{\cdot}(\mathbf{f}^n, M)$$
$$\begin{aligned} H^{\cdot}((\mathbf{f}), M) &= H^{\cdot}(C^{\cdot}((\mathbf{f}), M)) \\ &= \varinjlim_{n} H^{\cdot}(\mathbf{f}^n, M), \end{aligned}$$

where the last equality follows from the fact that the functor $H^{\cdot}$ commutes with limits.

2. The Čech cohomology

For this section we adopt the following notation. X will denote a scheme, $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. $A = \Gamma(X, \mathcal{O}_X), M = \Gamma(X, \mathcal{F})$; $\mathbf{f} = (f_i)_{1\leq i\leq r}$ a finite system of elements of A. $U_i = X_{f_i}$ is the open set of points $x \in X$ such that $f_i(x) \neq 0$; $U = \cup_i U_i$; and $\mathfrak{U}$ the open covering $(U_i)_{1\leq i\leq r}$ of U.

Suppose that X is noetherian or that the underlying topological space is quasi-compact. Then $\Gamma(U_i, \mathcal{F}) = M_{f_i}$. We set

$$U_{i_0 i_1 \cdots i_p} = \bigcap_{0\leq k\leq p} U_{i_k} = X_{f_{i_0} f_{i_1} \cdots f_{i_p}};$$

we have thus

$$\Gamma(U_{i_0 i_1 \cdots i_p}, \mathcal{F}) = M_{f_{i_0} f_{i_1} \cdots f_{i_p}}.$$

One can think of $M_{f_{i_0} f_{i_1} \cdots f_{i_p}}$ as the direct limit $\varinjlim_{n} M^{(n)}_{i_0 i_1 \cdots i_p}$ where $M^{(n)}_{i_0 i_1 \cdots i_p} = M$, via the homomorphisms $\varphi_{nm} : M^{(m)}_{i_0 i_1 \cdots i_p} \to M^{(n)}_{i_0 i_1 \cdots i_p}$ which is multiplication by $(f_{i_0} f_{i_1} \cdots f_{i_p})^{n-m}$ for $m \leq n$.

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Let $C^p_n(M)$ denote the set of alternating maps from $[1 \cdots r]^{p+1}$ to M. The *alternating* Čech p*-cochains* is the limit

$$C^p(\mathfrak{U}, \mathcal{F}) = \varinjlim_n C^p_n(M).$$

But in our notation we have that $C^p_n(M)$ is identified with $\mathbf{K}^{p+1}(\mathbf{f}^n, M)$ and the maps φ_{nm} are identified with $\varphi_{\mathbf{f}^{n-m}}$ defined in the last section. We thus have a canonical isomorphism that is functorial in $\mathcal{F}$

$$C^p(\mathfrak{U}, \mathcal{F}) \cong C^{p+1}((\mathbf{f}), M).$$

The isomorphism maps being compatible with the coboundary maps we get the following theorem:

Theorem 2.1. *If X is a noetherian scheme or its underlying topological space is quasi-compact, then there is a canonical isomorphism that is functorial in $\mathcal{F}$*

$$H^p(\mathfrak{U}, \mathcal{F}) \cong H^{p+1}((\mathbf{f}), M)$$

for $p \geq 1$. The following exact sequence is also functorial in $\mathcal{F}$

$$0 \to H^0((\mathbf{f}), M) \to M \to H^0(\mathfrak{U}, \mathcal{F}) \to H^1((\mathbf{f}), M) \to 0.$$

Now if $\mathfrak{U}'$ indexed by J (say) and $\mathfrak{U}$ indexed by I are two coverings of U with the property that for each $U'_j \in \mathfrak{U}'$ there is a i such that $U'_j \subseteq U_i$ then we say that $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$. Thus we can choose a map $\phi : J \to I$ such that $U'_j \subseteq U_{\phi(j)}$. This yields maps $\rho_\phi : C^p(\mathfrak{U}, \mathcal{F}) \to C^p(\mathfrak{U}', \mathcal{F})$. These maps are compatible with the coboundary maps and we get induced maps on the cohomology

$$\check{H}^p(X, \mathcal{F}) = \varinjlim_{\mathfrak{U}} H^p(\mathfrak{U}, \mathcal{F}).$$

Corollary 2.2. *Suppose that X_{f_i} are quasi-compact and there exist $g_i \in \Gamma(U, \mathcal{F})$ such that*

$$\sum_i g_i(f_i \mid U) = 1 \mid U.$$

Then for all quasi-coherent $(\mathcal{O}_X \mid U)$-modules $\mathcal{G}$, we have $H^p(\mathfrak{U}, \mathcal{G}) = 0$ for $p > 0$; if in addition $U = X$, the canonical homomorphism $M \to H^0(\mathfrak{U}, M)$ is bijective.

Proof : Since the $U_i = X_{f_i}$ are quasi-compact, the same is true of U. We can thus reduce to the case $U = X$. Then by Corollary (II.2.2) $H^p((\mathbf{f}), M) = 0$. The other claim follows directly from the above theorem. □

3. Cohomology of Affine schemes

The definition of the Čech cohomology groups as direct limits of coverings is unwieldy. The following important theorem of Leray gives us a handle on the cohomology groups in a particular case.

Theorem 3.1. *If the covering $\mathfrak{U}$ is acyclic for the sheaf $\mathcal{F}$ in the sense that $H^q(U_{i_1} \cap U_{i_2} \cap \cdots \cap U_{i_p}, \mathfrak{F}) = 0$ for $q > 0$ and any $i_1, \cdots, i_p$, then*

$$H^\cdot(\mathfrak{U}, \mathcal{F}) \cong \check{H}^\cdot(X, \mathcal{F}) \cong H^\cdot(X, \mathcal{F}).$$

Here $H^\cdot(X, \mathcal{F})$ is the sheaf cohomology groups to be defined later.

Theorem 3.2. *Let X be an affine scheme. Then for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ $H^p(X, \mathcal{F}) = 0$ for all $p > 0$.*

Proof : Let $\mathfrak{U}$ be a finite covering of X by the open affines $X_{f_i} = D(f_i)$ for $1 \leq i \leq r$. In this case the ideal generated by the f_i is equal to $A = \Gamma(X, \mathcal{O}_X)$. Thus by Corollary 2.2 we have that $H^p(\mathfrak{U}, \mathcal{F}) = 0$ for $p > 0$. Since this is true for any arbitrary such covering, we have by the definition of the Čech cohomology that $\check{H}^p(X, \mathcal{F}) = 0$ for $p > 0$. But this argument can be applied to X_f and also $X_f \cap X_g = X_{fg}$. Thus by Theorem 3.1 we have that $H^p(X, \mathcal{F}) = 0$ for $p > 0$. □