GROTHENDIECK TOPOLOGIES

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Here we begin the study of étale cohomology. The material here is from Deligne's SGA $4\frac{1}{2}$.

1. Sieves

Let X be a topological space. A set $\mathfrak{U}$ of open subsets of X is said to be a *sieve* if for every $U \in \mathfrak{U}$ and $V \subseteq U$ an open set, we have $V \in \mathfrak{U}$. A sieve is said to be a *covering* if the union of the sets it contains is X. If $\{U_i\}$ is a family of open subsets of X, the sieve *generated by the family* $\{U_i\}$ is by definition the set of open subsets of X that are contained in some U_i.

Suppose $P(U)$ is a property defined for open subsets U of X. Such a property is said to be *local* if for every covering sieve $\mathfrak{U}$ of every open set U of X, $P(U)$ is true iff $P(V)$ holds for each $V \in \mathfrak{U}$.

2. Grothendieck Topologies

Let $\mathcal{S}$ be a category and let U be an object of $\mathcal{S}$. We call a subset $\mathfrak{U}$ of objects of $\mathcal{S}/U$ (i.e. the category whose objects are objects of $\mathcal{S}$ together with a map to U, and morphisms are the ones that commute with this extra structure) a *sieve over* U if the following property holds: If $\varphi : V \to U$ is in $\mathfrak{U}$ and $\psi : W \to V$ is a morphism in $\mathcal{S}$ then $\varphi \circ \psi : W \to U \in \mathfrak{U}$.

If $\{\varphi_i : U_i \to U\}$ is a family of morphisms, the sieve over U *generated* by this family is by definition the set of morphisms $\varphi : V \to U$ that factor through one of the morphisms in the family.

Let $\mathfrak{U}$ be a sieve over U and let $\varphi : V \to U$ be a morphism. The restriction of the sieve to V denoted by $\mathfrak{U}_V$ is by definition the sieve over V generated by the maps $\psi : W \to V$ such that the composite $\varphi \circ \psi : W \to U$ is in $\mathfrak{U}$.

Definition 2.1. Let $\mathcal{S}$ be a category. To give a *Grothendieck topology* on $\mathcal{S}$ is to give for every object U in $\mathcal{S}$ a set $C(U)$ of sieves over U said to be *covering sieves* over U, such that the following properties hold:

(1) The sieve generated by the identity morphism of U is covering.
(2) If $\mathfrak{U}$ is a covering sieve and if $\varphi : V \to U$ is a morphism then $\mathfrak{U}_V$ is also a covering sieve.
(3) A locally covering sieve is covering. This means that if $\mathfrak{U}$ is a covering sieve of U and $\mathfrak{U}'$ is a sieve such that for every $V \to U$ coming from $\mathfrak{U}$ the restriction $\mathfrak{U}'_V$ is covering, then $\mathfrak{U}'$ is also covering.

A category that is given a Grothendieck topology is called a *site*.

Given a site $\mathcal{S}$, we call a pre-sheaf on $\mathcal{S}$ a contravariant functor $\mathfrak{F}$ from $\mathcal{S}$ to the category of sets. If U is an object of $\mathcal{S}$, we call elements of $\mathfrak{F}(U)$ sections. For all morphisms $V \to U$ and all $s \in \mathfrak{F}(U)$, we denote by $s|_V$ the image of s in $\mathfrak{F}(V)$.

Let $\mathfrak{U}$ be a sieve on U, a section of U is said to be given $\mathfrak{U}$-locally if for all $V \to U$ in $\mathfrak{U}$ we are given $s_V \in \mathfrak{F}(V)$ such that for all morphisms $W \to V$, we have $s_V|_W = s_W$. We say that $\mathfrak{F}$ is a *sheaf*, if for all objects U of $\mathcal{S}$, for all covering sieves $\mathfrak{U}$ of U and all sections given $\mathfrak{U}$-locally $\{s_V\}$, if there is a unique section $s \in \mathfrak{U}$ such that

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$s|_V = s_V$, for all $V \to U$ contained in $\mathfrak{U}$. Changing the category of sets in the definition to abelian groups we get *abelian sheaves.* One can show that the category of abelian sheaves on $\mathcal{S}$ is an abelian category with enough injectives. A sequence $\mathfrak{F} \xrightarrow{f} \mathfrak{G} \xrightarrow{g} \mathfrak{H}$ of sheaves is exact if, for every object U of $\mathcal{S}$ and all $s \in \mathfrak{G}(U)$ such that $g(s) = 0$ there exists locally a t such that $f(t) = s$; i.e., there exists a covering sieve $\mathfrak{U}$ of U such that for all $V \in \mathfrak{U}$, a section t_V of $\mathfrak{F}(V)$ such that $f(t_V) = s|_V$.

Example 2.2. Let X be a topological space, and $\mathcal{S}$ a category whose objects are open subsets of X and morphisms are inclusions. We define a Grothendieck topology on $\mathcal{S}$ by defining the covering sieve for each U to be a sieve that covers U. Then the Grothendieck topology matches the usual topology and the category of sheaves on $\mathcal{S}$ is equivalent to the usual category of sheaves on X.

3. Cohomology

Suppose that the category $\mathcal{S}$ contains a final object X. Given an abelian sheaf $\mathfrak{F}$ we call the sections in $\mathfrak{F}(X)$ *global sections* and denote it $H^0(X, \mathfrak{F})$ or $\Gamma\mathfrak{F}$. The (global sections) functor $\mathfrak{F} \to \Gamma\mathfrak{F}$ is left exact and we denote the i-th cohomology group to be i-th right derived functor of the global section functor.