ABELIAN CATEGORIES - III

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1. Injective Objects (contd.)

Let $\mathbb{C}$ be an abelian category. We will find it convenient to impose the following two axioms at times:

(AB3) For every family $(A_i)_{i\in I}$ of objects of $\mathbb{C}$ the direct sum

$$\bigoplus_{i\in I} A_i$$

exists.

(AB5) The axiom (AB3) holds and if $(A_i)_{i\in I}$ is a filtered increasing family of sub-objects of $A \in \mathbb{C}$. B any sub-object of A, then we have

$$\left(\sum_i A_i\right) \cap B = \sum_i (A_i \cap B).$$

Here $\sum A_i$ denotes the sup of the A_i, and $P \cap Q$ the inf of the sub-objects P and Q of A.

The axiom (AB5) can also be expressed as follows: (AB3) holds, and if $A \in \mathbb{C}$ is the sup of an increasing family of sub-objects of A_i, and if for each i we are given a morphism $u_i : A_i \to B$ such that for $A_i \subseteq A_j$, u_i is induced by u_j, then there is a morphism $u : A \to B$ induced by the u_i.

Recall the definition of an injective object. Let $\mathbb{C}$ be an abelian category and $M \in \mathbb{C}$. M is said to be *injective*, if given a map from $u : B \to M$ where B is a sub-object of A, there is a map $u' : A \to M$ that extends u.

Our object is to prove the following lemma, which is easily seen as a generalization of Theorem 3.1 in Abelian Categories II.

Lemma 1.1. *Let $\mathbb{C}$ be an abelian category satisfying (AB5) and has a generator U. Then an object $M \in \mathbb{C}$ is injective iff for every sub-object V of U, and all morphisms $v : V \to M$, v extends to a morphism of $U \to M$.*

Proof :(Sketch) It clearly suffices to show that the condition is sufficient. Let B be a sub-object of A and $u : B \to M$ a map. We need to show that there is a u' making the following diagram commutative:

$$\begin{tikzcd} B \arrow[r] \arrow[d, "u"] & A \arrow[dl, dotted, "u'"] \\ M & \end{tikzcd}$$

We will follow the basic outline of the proof of Theorem 3.1. Consider the set P of extensions of u to a sub-object of A containing B (since U has a generator the sub-objects of A do indeed form a set). Give this set an ordering based on extension of u. By the axiom (AB5) and the comments following it, this set is inductive i.e., every increasing sequence has a maximal element. Thus by Zorn's lemma, P has a maximal element. Thus we have reduced to the case that B itself is a maximal extension of u. We will show that if

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$B \neq A$ then there is an extension of u to a $B' \neq B$.

Let $j : U \to A$ such that $j(U) \not\subseteq B$. Set $B' = j(U) + B$ thus $B' \neq B$. Let $V = j^{-1}(B)$ the inverse image of B under j. Let $j' : V \to B$ be the morphism on B induced by j. Consider the sequence of morphisms

$$V \xrightarrow{\varphi'} U \times B \xrightarrow{\varphi} B' \longrightarrow 0$$

where φ' is inclusion on the first component and $-j$ on the second, and $\varphi = j \circ \pi_1 + 1 \circ \pi_2$ where π_i are the projections of $U \times B$. This turns out to be an exact sequence. Thus to define a morphism $v : B' \to M$ it is enough to define a morphism $w : U \times B \to M$ such that $w \circ \varphi' = 0$. Let k be an extension of $uj : V \to M$ to U and take for $w : U \times B \to M$ the morphism whose components are k and u. We see then that $w\varphi' = 0$ and that the morphism $v : B' \to M$ defined by w extends u. This proves the lemma. $\square$